Modeling population dynamics of rhino-poacher interaction across South Africa and the Kruger National Park using ordinary differential equations

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Abstract

In this thesis, a system of ordinary differential equations (ODES) is presented to model the population dynamics between poachers and rhino as a predator-prey system in both South Africa (SA) and the Kruger National Park (KNP). The data used in this thesis consists mainly of government and police reports, as well publications from several NGOs and the limitations caused by this lack of applicable data are explored. The system dynamics are based on Lotka-Volterra differential equations, which are extended to include both a carrying capacity and the Allee effect. This thesis parameterises a model of the dynamics of the interaction between rhino and poachers for some time t and makes predictions based on the interpolation of the available data. The unknown rates and parameters relating to the behaviour of populations R and P are optimised by initially using a combination of educated guesses made from the available data or trial and error until set values are obtained. The remaining unknowns are numerically optimised based on the fixed value parameters. This is considered a constrained system, and the results obtained can only be viewed as constrained predictions based on parameter values obtained by a combination of trial and error and numerical optimisation; namely root mean square (RMS) error considering the available data and model solution at time t. Those parameter values obtained through RMS are regarded as error-minimising parameters within the scope of this research, and make up the final models which are referred to as the models which have been fitted to data. This thesis is an introductory, exploratory work into future attempts at modeling population dynamics with very little or no available data. The models are solved for in a constrained system, limiting the resulting predictions to constrained estimates based on the assigned values to unknown parameters. These solutions predict rhino stabilisation for both models, with active poachers dying out in the KNP but general co-existence observed across SA, within the constrained system.

Keywords: Rhino poaching, mathematical modeling, Lotka Volterra, differential equations

1 Introduction

1.1 Aims

This research aims to mathematically model a real system of rhino poaching using two systems of ordinary differential equations (ODEs), representing the South African (SA) and the Kruger National Park (KNP) rhino-poacher interactions, in accordance with the limited available data. These ODEs are solved and produce results which represent possible future scenarios of the considered populations, based on the available rhino population data. The results produced in this research are viewed as constrained predictions. The models presented in this research are considered an initial and exploratory work into modeling rhino-poacher dynamics using this approach.

For each model a pair of ODEs are presented as a predator-prey system. In this research the predators are the active poachers, P and the prey are the rhino, R. These ODEs are constructed around the available data, discussed below.

1.2 Data

All of the available population data regarding rhino poaching across South Africa are presented here. The SA and KNP rhino population figures are considered in Sections 1.2.1 and 1.2.2 respectively. The only data available on poachers in SA are presented in Section 1.2.3 alongside the corresponding number of rhino poached. All data are annualised, however there is a lack of uniformity. Due to security concerns and the inconsistent nature of collecting rhino population data, there are some years during which the population figures are not made available.

1.2.1 South African Rhino Population

Rhino population data is difficult to obtain and this is attributed to a lack of effective survey and observation techniques as well as various security concerns. An estimated population table for both the black and white rhino in SA during the period of $1993 - 2010$ [1] is given below:

	White Rhino	**Black Rhino**	**Total**
1993	6376	897	**7273**
1995	7095	962	**8057**
1997	7913	1043	**8956**
2005	13555	1384	**14939**
2006	15941	1456	**17397**
2007	16273	1512	**17785**
2008	18352	1587	**19939**
2009	19409	1678	**21087**
2010	18780	1916	**20696**

Table 1: South African rhino population estimates for 1993-2010 [1].

Rhino population data obtained from several different sources [2, 3, 4] are used for comparison against those obtained in [1]:

	White Rhino	Black Rhino	Total
2007	16273	1488	**17761**
2010	18796	1915	**20711**
2012	18933	2068	**21001**
2015	18413	1893	**20306**

Table 2: South African rhino population estimates for 2007, 2010, 2012 and 2015 [3, 4].

The ecological carrying capacity (ECC) is the maximum species population size that an ecosystem can sustain. For total rhino in South Africa, the carrying capacity is estimated at between $26000 - 28000$ rhino, based on a rough guide that population numbers are exceeding 75% of ECC [2] in terms of the total rhino population counts from $2010 - 2015$ as presented in Table 2 above. These numbers are regarded as very rough estimates since the ecosystems across South Africa have a large range of types, however for illustrative and mathematical modeling purposes in this research the carrying capacity for rhino in South Africa is set at $K = 26000$.

1.2.2 KNP Rhino Population

The KNP is the most densely populated SANPark in South Africa. Table 3 below shows the number of rhino surviving the first year of their adolescent life compared to the number of rhino poached in the same year:

	Births: White Rhino	**Poached:** White Rhino	**Births:** Black Rhino	**Poached:** Black Rhino
2013-2014	854-992	745	18-26	17
2014-2015	725-810	818	29-42	52

Table 3: KNP rhino birth and poaching-related death estimates for 2013-2015 [5]

Illegal poaching of rhino occurring per province in South Africa for the years $2015 - 2017$ [6] are summarised in the table below:

	EC	**FS**	**GT**	**NL**	**LP**	**MP**	**NC**	**NW**	**WC**	**KNP**
2015	14	10	1	116	91	67	2	46	1	826
2016	17	17	6	162	90	32	12	56	0	662
2017	12	38	4	222	71	49	24	96	0	504

Table 4: South African rhino poaching estimates per province for 2015-2017 [6]

The included provinces are the Eastern Cape (EC), Free State (FS), Gauteng (GT), Kwa-Zulu Natal (NL), Limpopo (LP), Mpumalanga (MP), Northern Cape (NC), North West (NW), Western

Cape (WC) and the Kruger National Park (KNP) is considered separately. Table 4 shows the highest number of rhino deaths from poaching take place in the KNP.

Rhino deaths from poaching in the KNP alone during 2017 are higher than the total country-wide poaching numbers over the period $2000 - 2009$. The 662 rhino poached in the same area from 2016 alone are only 20 fewer rhino than the total country-wide poaching deaths over the period $2000 - 2010$. This thesis considers the KNP exclusively.

1.2.3 Poacher Data

The South African poaching-related arrest figures are correlated from several reports [7, 8, 9, 10] for the years $2010 - 2017$ inclusive and are presented below:

	Rhino Poached	**Poacher Arrests**
2000	7	-
2001	6	-
2002	25	-
2003	22	-
2004	10	-
2005	13	-
2006	24	-
2007	13	-
2008	83	-
2009	122	-
2010	333	165
2011	448	232
2012	668	267
2013	1004	343
2014	1215	386
2015	1175	317
2016	1054	680
2017	1028	518

Table 5: South African rhino poached and poacher arrest estimates for 2000-2017 [6, 7, 8, 9, 10]

The final data sets used in this research are discussed below.

1.3 Final Data Sets

Here only the sets of data used in the thesis are presented. Poacher data is given in Table 5 and is not repeated here.

1.3.1 South Africa

The South African rhino population figures used for the SA rhino-poacher model are summarised below:

	White Rhino	Black Rhino	Total
2005	13555	1384	**14939**
2006	15941	1456	**17397**
2007	16273	1488	**17761**
2008	18352	1587	**19939**
2009	19409	1678	**21087**
2010	18796	1915	**20711**
2012	18933	2068	**21001**
2015	18413	1893	**20306**

Table 6: South African rhino population estimates for 2005-2015

1.3.2 Kruger National Park

The data considered in this thesis relating to the Kruger National Park are implemented in the KNP rhino-poacher model, and presented below:

	White Rhino	Black Rhino	Total
2012	10495	458	**10953**
2013	8968	414	**9382**
2014	8691	309	**8928**
2015	8875	308	**9239**

Table 7: KNP rhino population estimates for 2012-2015 [5]

This concludes the data used in the modeling process of this thesis. The aim of collecting the raw data is to compare the results obtained in this thesis to the available time series.

2 Problem Statement

Rhino poaching has become one of the main causes of death and rhino population decline since the early 2000s. As discussed, data pertaining to rhino poaching and rhino population figures are very difficult to obtain. These data mostly focus on population numbers for rhino, and show that there are currently between $28353 - 30190$ rhino left in the world [11] as opposed to the 500000 rhino that populated Africa alone at the start of the 20^{th} century [12]. The table below shows the five remaining rhino species and their endangered status [11]:

Species	Remaining Population	Endangered Status
Javan Rhinoceros	67	Critically Endangered
Sumatran Rhinoceros	< 80	Critically Endangered
Black Rhinoceros	$5040 - 5458$	Critically Endangered
Greater One-Horned Rhinoceros	3500+	Vulnerable
White Rhinoceros	$19666 - 21085$	Near Threatened

Table 8: Current status of the rhinoceros species [11]

There are three categories of threatened species: vulnerable, endangered and critically endangered [13], as well as five criteria used to decide whether (and to what extent) a species is threatened [13]. These criteria are:

1. Declining population either past, present and/or projected,
2. Geographic range size, decline or variance,
3. Small population size, decline or variations,
4. Very small population or very restricted distribution, and
5. Quantitative analysis of extinction risk.

In order for a species to be listed as critically endangered, one of the five categories the species falls under must exhibit extreme behaviour [13]. For example, considering category 1, for a species to become critically endangered there needs to be a $\geq 80\%$ of "population reduction observed, estimated, inferred or suspected in the past where the causes of reduction may not have ceased or may not be understood or may not be reversible" [13].

Africa is home only to the black and white rhinoceros. Table 8 shows 60% of the current world rhino species is critically endangered and several sub-species have already gone extinct, namely the Western black rhino in Cameroon [14], the Indochinese Javan rhino in Vietnam [14] and the northern white rhino in Kenya [15]. The reason for the rhino species' current critical endangerment has been attributed heavily to illegal poaching.

The illegal worldwide rhino horn trade market has been central in this crisis as the only trade-off criminals face for exuberant amounts of black market money is potential prosecution. Poachers reportedly saw returns upwards of 65000 US Dollars per kilogram of rhino horn ivory in 2012 [16].

2.1 Thesis Outline

The prior literature is reviewed in Section 3. The methodology implemented in prior research is considered, as is the representation of poaching.

In Section 4 the rhino-poacher models are formulated. Initially analysis of general dynamic systems is performed to explore the available mathematical application to the dynamics of rhino and active poacher populations. The assumptions which govern the system of differential equations aim to represent well the dynamics of their populations, namely the rhino and active poachers. These

assumptions are the logical guidelines that are used to shape the model and justify the selection of specific parameters as well as their respective values or ranges. A Lotka-Volterra system of differential equations are then explored and expanded on to include initially the carrying capacity and, upon iteration, the Allee effect. The limitations based on the limited data are presented, after which the new rhino-poacher system and its related parameters are introduced. The South African (SA) and Kruger National Park (KNP) models are formulated and discussed, after which their respective parameter values are found through optimisation, initially through an intuitive process of variation across the ranges of certain parameters until a suitable fit (as judged by eye) is achieved, after which the remaining parameters are numerically optimised using root mean square (RMS) error. The limitations of the model are then discussed.

The SA and KNP models fitted to the data and their results are presented in Section 5. Initially the parameters obtained through RMS, referred to in this thesis as "error-minimised" parameters, are given along with their respective system of ordinary differential equations (ODEs). These ODEs are then solved for, and the solutions plotted after which stability analysis is performed on the system.

The thesis conclusions are discussed in Section 6

Potential future model improvements are considered in Section 7 to show what can be improved on from this thesis.

3 Literature Review

Studies within the field of population modeling focus on population dynamics of certain species of animal. In the scope of this thesis the focus is on a species facing rapid population decline and possible extinction due to a host of reasons, namely the rhino. Prior research aimed at solving problems relating to the rhino population has exclusively focused on population density and the effects that seasonal change and unpredictability have in the variance of the rhino population. Research was performed on different scales, mostly focusing on smaller and more isolated regions like national parks rather than entire countries. National parks are likely to keep information relating to their endangered population numbers and density coefficients, in some cases even rhino gender and age are available metrics either offered by the national parks or obtained by/from those working within them.

Prior research is discussed here, in order to highlight the possible approaches as well as comment on the strengths and weaknesses of each of the models.

Both Cromsigt, J. P., et al. [17] and Thuo, D. N., et al. [18] considered black rhino populations in national parks, the former in South Africa and the latter in Kenya. These two studies consisted of rhino and weather data from a small area with sets of population and translocation data occurring before 1999. Ferreira, S. M., et al. [19] made use of more recent population data for both black rhino (2008) and white rhino (2010) in order to predict the 2013 population size and poaching rates in the Kruger National Park (KNP). Although collectively each of these studies has focused on a national park, the KNP is the largest of these and is also affected by poaching. Aldila,

D., et al. [20] considered the effect translocation has on the black rhino population across South Africa based on the 2011 national black rhino population figures. The research split the considered black rhino population into three separate populations groups, one of which is designated as a donor population. Their goal is to “maximize the growth rate of all three populations" [20] and a donor population is seen as a controlled group of rhino from which either male or female rhino are removed when another population group is not showing the anticipated growth. The rhino from the donor population go to the population group which is known as the receiver population as this group receives rhino from the donor population through translocation.

3.1 Methodology

In this section the approaches taken in prior research are discussed and compared.

The research in [18] is performed using VORTEX (Ver. 7) for all stochastic population simulations, using data pre-dating 1998. Vortex PVA software [21] is provided under a CreativeCommons Attribution-NoDerivatives International License, courtesy of the Species Conservation Toolkit Initiative (https://scti.tools). This software is not discussed in [18] but [21] describes VORTEX as "an individual-based simulation of deterministic forces as well as demographic, environmental and genetic stochastic events on wildlife populations. It can model many of the extinction vortices that can threaten persistence of small populations." [21]. The population simulation goes through several iterations to observe multiple possible population fates [21] and VORTEX models discretely population dynamics as sequential events occurring in accordance with random variable probabilities following distributions specified by the user [21]. The data set used in [18] is made up of gender, age and reproduction figures which included observed blacked rhino mortality and birth numbers. The research in [18] performed simulations in VORTEX v7 and these are assumed under density independence and the exclusion of possible inbreeding due to large enough population numbers in the area of consideration. Carrying capacity is decided based on the resources available to sustain maximum possible black rhino population in the relative study area [18]. Computed rates included mortality, survival and reproductive rates, arranged by age, and these are used in six simulations, during which the only varied rates are those with environmental variation and translocation rates, also known as harvesting [18]. Environmental variation represented the possibilities of drought and the presence of other herbivorous large animals and predatory lions which affect the black rhino population and its growth rate [18]. This research does not disclose the model used but instead provide the results of the six performed simulations, each with different translocation rates, translocation periods and environmental variation and how these cause change in the predicted population sizes.

The data collected for black rhino in the uMkhuze and HluhluweâiMfolozi National Parks [17] made predictions on the fluctuations of the respective black rhino populations through the use of a mathematical model. The modeling approach is simplistic and based on the following:

$$P(t) = N(t) + \epsilon(t), \tag{3.1}$$

an equation originally presented in "*How many rhinos are there in the Hluhluwe-Umfolozi Game Reserve? Unpublished paper, ORSSA Conference, Illovo Beach, South Africa*" by Fatti and Corrigall in 1997 and used as a basis upon which to expand on [17]. This considers the observed census population number at time t, $P(t)$, and the model population number at time t, $N(t)$. As the

observed population has some variance from the model population, the error rate $\epsilon(t)$ represents all the factors which affect the rhino population. The research performed by Cromsigt, J. et al. [17] aim to predict population numbers for the following year by implementing a simple model, as shown in equation (3.1), which give insight into animals who can potentially be moved out of the population [17]. Following this they used various models to evaluate what impact minor variation in model structure had on their model results, and five models are tested which focus mainly on the density dependence structure [17]. The considered equations are listed below concluding with the Fowler translocation model and beginning with the exponential model:

$$N_{t+1} = N_t + rN_t.$$

This model states that a population N at time $t+1$ is equivalent to the population N at time t multiplied by some parameter r. Since there is no way to prevent or curb the population increase for positive r, this is known as the exponential model. The following model improves on this:

$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right), \tag{3.2}$$

where K is introduced as the carrying capacity, and the term $(1 - \frac{N_t}{K})$ is introduced as density dependence turns into the logistic model. The fraction $\frac{N_t}{K} \in [0, 1]$ assuming the population N can reach either 0 or K. Equation (3.2) becomes the logistic translocation model when including the parameter for harvesting, or translocation, h. This equation is shown below:

$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right) - h. \tag{3.3}$$

Equation (3.2) becomes what is referred to in [17] as the Fowler model when considering density dependence as non-constant, namely $n \neq 1$ as shown below:

$$N_{t+1} = N_t + rN_t\left(1 - \left(\frac{N_t}{K}\right)^n\right). \tag{3.4}$$

This is the Fowler model. Including the constant for translocation, or harvesting h, equation (3.4) becomes:

$$N_{t+1} = N_t + rN_t\left(1 - \left(\frac{N_t}{K}\right)^n\right) - h, \tag{3.5}$$

which is the Fowler translocation model.

Each of these iterations are chosen to test the fit of the population number (model solution) against the observed census population number (available data). The best estimates achieved in [17] for their initial population size $N(0)$, along with the various parameters of the model which allowed them to make predictions on $N(t)$ [17]. The solutions of $N(t)$ are obtained by minimising:

$$\sum_{t=1}^{n} \epsilon(t)^2. \tag{3.6}$$

With the estimated values of $N(0)$ and the model parameters, the best estimates for population size N at times $t \in [1, n]$ are then computed. This is repeated for each iteration to observe what effect the variation of structure in every model had on its results. They concluded that the best fits for the game reserves considered in this research are found with the Fowler model as shown in

equation (3.4).

Aldila, D. et al. [20] propose a deterministic model to predict the effects a three-population black rhino translocation system has on the black rhino population of South Africa. The population data used in [20] is as recent as the black rhino figures of 2011 and the differential equations making up their model model are given below:

$$\frac{dy_1}{dt} = \mu_1 y_1(t)\left(1 - \frac{y_1(t)}{k_1(t)}\right) - h_1(t)y_1(t) \tag{3.7}$$

$$\frac{dy_2}{dt} = \mu_2 y_2(t)\left(1 - \frac{y_2(t)}{k_2(t)}\right) + \rho h_1(t)_1(t) - h_2(t)y_2(t) \tag{3.8}$$

$$\frac{dy_3}{dt} = \mu_3 y_3(t)\left(1 - \frac{y_3(t)}{k_3(t)}\right) - (1-\rho)h_1(t)y_1(t) + h_2(t)y_2(t) \tag{3.9}$$

where μ_i, the recruitment rate for each population, is the difference between birth rate and death rate, at very low values of $y_i(t)$. The translocation rate is given by $h_i(t) \in [0,1]$ and $\rho \in [0,1]$ represents how desperately a population needs rhino. The three populations $y_i(t)$ where $i \in [1,3]$ considered in [20] are the donors $y_1(t)$, the primary receivers $y_2(t)$ and the secondary receivers $y_3(t)$. The carrying capacity is $k_i(t) = k_{0i} + a_i sin(\frac{\pi t}{b_i})$ where k_{0i} is initial carrying capacity for i, a and b are constants to account for fluctuations of k. This also accommodates for rainfall variability and environmental geography as k is a function of time.

Here every population is decreased at the translocation rate h_i, and this is the only way in which the populations of y_i decrease. This system assumes no mortality rates but do have a 'birth rate' μ_i, as $y_i \leq k_i$ for each case. This system therefore states these populations decrease due to the translocation removal of rhino from each of the three populations. They show human intervention and translocation have an impact when implemented correctly, and have also considered rhino size sex and age as well as the climate variability (within their definition of k_i). The act of taking rhino out of one of the three populations (harvesting) is performed without harm to the rhino. The main objective is to optimise black rhino population growth through translocation noting the donor population is not affected negatively and the receiving population must benefit from the harvest.

Disruption of rhino demography by poachers may lead to population declines in the Kruger National Park. An exponential function model is considered in [19] to predict black rhino population sizes and poaching rates in the Kruger National Park in 2013. This research is preceded in [22] with the same methodology used to predict white rhino population sizes and poaching rates for 2012. Both papers focused on the KNP [19, 22] and used a combination of rhino removal figures, number of rhino poached, rhino population figures (made up of surveys and estimates) and the presence of rhino across various landscapes in the KNP [19, 22]. These figures are then used to illustrate occurrences of poaching growing exponentially, and for the black rhino [19] this equation is given below:

$$y = 0.002e^{0.633x}e^{-0.074x},\ r^2 = 0.96, \tag{3.10}$$

and is simplified to:

$$y = 0.002e^{0.559x},\ r^2 = 0.96. \tag{3.11}$$

The exponential equation representing white rhino poaching [19, 22] is shown below:

$$y = 0.042e^{0.616x},\ r^2 = 0.89. \tag{3.12}$$

It is from these exponential equations that predictions of a decline of 22 black and 982 white rhino in the KNP in 2013 were made [19]. The figures listed in Table 3 show that during the $2013-2014$ annual survey there were 17 black and 745 white rhino poached in the KNP. Ferreira, S., M. et al [19] have therefore overestimated these figures by 30% in their model's results.

This model is designed to use data obtained from surveys to generate solution curves which relate closely to poacher behaviour observed over the period of $1998-2010$ in order to better and more accurately predict those of 2013. Although the results are overestimates the model still produces results which display correlation to the real world figures.

3.2 Representation of Poaching

This research considers the poacher-rhino interaction as a predator-prey system, and both populations are considered separately and independently of one another when modeling this system. Here the prior research conducted is discussed in terms of the representation of poachers or poaching.

Neither Cromsigt, J. P., et al. [17] nor Thuo, D. N., et al. [18] consider poachers or poaching rates within their research. The simple reason is that the data used in their research relates only to population size and translocation figures from $1988-1998$, one considering two game reserves [17] and the other a national park [18]. With no rate of population decline due to poaching, the rhino population in these models is decreased by removal of individual rhino due to translocation: the process of removing rhino from one habitat in order to repopulate another. In these scenarios, mainly due to the time frame considered, it is understandable why no poacher or poaching rates are included as the poaching mortality rates of those times were almost non-existent compared to those of the 21^{st} century. The Hluhluwe-iMfolozi Park experienced an "avalanche" of poaching since 2016 [23], citing the decrease of Kruger National Park poaching activities as a catalyst for this [23]. In 2017 a total of 222 rhino were poached in the province of Kwa-Zulu Natal, with 85% of that figure ($\pm$188 rhino) being poached in the Hluhluwe-iMfolozi Park [23].

Translocation is discussed by Aldila, D., et al.[20] in terms of the black rhino population of South Africa. There is no representation of poaching effect on the rhino population within this model even though this research has provided evidence of the impact poaching has had on South Africa's black rhino population. The population decrease is again represented by some rate of this population, used to represent translocation.

On the other hand, Ferreira, S. M., et al. [19] consider the effect poaching has on the rhino population of the Kruger National Park as a rate in terms of the rhino population: some parameter α for a rhino population R to act as a percentage of the current population; αR. This imposes a limitation on the accuracy of the poacher-rhino interaction, as the population relating to the act of poaching is clearly interacting with the rhino population. Hence the chosen approach, though inclusive of the clear effect poaching has on the rhino population, does not consider the active poacher population. Poaching does not take place at a consistent rate relating to the number of rhino "available" for poaching, but instead on a collection of outside factors such as rhino horn price, rhino habitat security and available equipment, even the availability of poachers to cite a few. It is clear that the poaching rate depends on factors difficult to accurately account for, along with the rhino population. This research therefore considers the rate at which rhino leave the poacher-rhino system due

to poaching as a poacher-rhino interaction rate, $\alpha(R*P)$, where P is the active poacher population.

Poaching rates are only considered in prior research when poaching has a noticeable effect on the rhino population. The effects of omitting a tangible poaching population in each of the research papers is discussed below.

3.2.1 Inclusion of Poaching and Possible Improvements

Research in [17] and [18] is based on data for the period $1989-1998$. Although the current situation in the Hluhluwe-iMfolozi Game Reserves is worsening due to poaching [23] and the Kenyan rhino poaching rates spiked in 2013 [24] there is no evidence of such problems during the time frame considered in this research. Therefore the exclusion of poaching data does not have a noticeable impact on the representation of the real-world state at the time.

On the other hand Ferreira, S. M., et al. [19] considered the 2008 and 2010 black and white rhino population growth rate estimates respectively. During this time rhino poaching was becoming a more prominent problem for the rhino population, this clearly reinforced by the inclusion of a poaching rate within the research. This rate, however, only considers the poaching impact on the rhino population as a rate of this rhino population. This implies that poaching occurs at a constant rate relating to the total number of rhino present in a rhino habitat.

Aldila, D., et al. [20] proposed a deterministic model based on the same principles used in this research, and considered a general rhino population spread across two donor population and a receiver population in terms of rhino translocation. Their research focused on increasing the black rhino population through this process, citing the 2011 population figures as a starting point. This research was performed, then, during a period of high poaching rates across South Africa, and the exclusion of this important factor in their model suggests a research that did not consider explicitly a phenomenon easily observed within the poacher-rhino system. Solutions and conclusions made and reached without the acknowledgment of how poaching affects the practice of translocation can be considered as incomplete. Although translocation research only focuses on the act of removing rhino from one habitat to help increase the population of another, it is stressed that the right rhino needs to be selected [20] in order to avoid cases such as in-breeding, miscarriage or the inability of rhino to reproduce. Poaching occurring in these donation populations potentially causes chaos by removing the chosen rhino, and acknowledging this results in a broader understanding of the problem and its possible solutions.

Research on poaching is considered as it has clearly impacted the rhino population across South Africa and the various game reserves, national parks and private sanctuaries. Although the poaching rate has begun marginally decreasing over the course of the last few years, it still remains one of the largest rhino population increase deterrents and a difficult problem to solve. It is because of these observations that this research considers a separate active poacher population in order to represent the poacher-rhino interaction rate that occurs in the event of rhino poaching.

4 Rhino-Poacher Model

In this section, dynamic systems are introduced and discussed in order to lay the foundation for the remainder of this thesis. The assumptions relating to the rhino-poacher model are presented, after which the modeling process begins with a simple system of Lotka-Volterra equations [25]. The set of Lotka-Volterra equations is expanded to include the concepts explored in the discussion of dynamic systems, arriving at the system of differential equations used in this thesis. These models act as a foundation upon which to improve and further the research regarding rhino poaching in the scope of systems of differential equations.

4.1 Software Used

All non-trivial figures and results presented in this research from this point forward are obtained in Mathematica v11.1. Algorithm outlines are given in Sections 4.10.1 and 4.10.2 for all code used.

4.2 Dynamic System Analysis

A dynamic system consists of states for every time t under consideration, and each of these states are subject to change or evolve, thereby dictating which states follow the current (or initial) state [26]. The state in which the system finds itself, whether equilibrium, a fixation into oscillating cycles or chaotic fluctuation, the system's dynamics are what describe these events [25]. In the event of a stable system, this stability is a result of "forces acting in cohort to produce a balance of 3 tendencies" [26], meaning that in specific cases a minute disturbance is all that is necessary to shift the system into an entirely different and unique state, and this is referred to as bifurcation.

Natural phenomena are mathematically represented by discrete subsystems each with their unique set of internal forces [26]. Hence to avoid a system which is far too complex to represent mathematically, the phenomena under consideration are accounted for and simplified by the individual modeling the phenomenon. In this thesis, the phenomenon in question is rhino population variation due to poaching, hence the system constructed around this must represent exclusively that. The dynamical system under consideration in this thesis is represented by ordinary differential equations (ODEs).

Equilibrium is achieved by a system in which opposing forces are equal or balanced, resulting in a state of the system which stays constant and does not experience variation. Stability of a system is divided into local and global stability. A system displays local stability when the system state moves away from the equilibrium following a perturbation, such as a disruption in the considered climate, and then returns to equilibrium. For local minima this perturbation takes place within a defined range of the equilibrium for the system to return to equilibrium. Global stability, in contrast, dictates that a system state returns to equilibrium regardless how large the perturbation is. Simply put a system can have several local minima, but only one global minimum.

In order to represent a dynamical system using ODEs the variables and parameters which affect the behaviour of the observed phenomenon are chosen. These allow for the formation of governing equations. Here both a one- and two-dimensional system of differential equations are considered and analysed both geometrically and using local linearisation.

4.2.1 One-Dimensional System

Consider a general one-dimensional equation:

$$\frac{dR}{dt} = f(R), \tag{4.1}$$

where $f(R)$ is a continuously differentiable function of R. Ffor $f(R) > 0$ the population size R increases, and for $f(R) < 0$ it decreases. The point $R = R^*$ is referred to as an equilibrium point, alternatives include critical point, fixed point, steady state or rest point at which the following equation is satisfied:

$$\left.\frac{dR}{dt}\right|_{R=R^*} = 0, \tag{4.2}$$

which is summarised by: a fixed point is that point $R = R^*$ for which the rate $\frac{dR}{dt} = 0$. These fixed points are solved for by setting:

$$f(R^*) = 0. \tag{4.3}$$

Several values R^* can satisfy equation (4.2), meaning there is no change to the system at these fixed points as time t increases. In [26] the important basic principles of fixed points are discussed, namely that in addition to non-zero fixed points that may be reached by population R, the trivial equilibrium is generally located at $R^* = 0$ which reinforces the law that populations cannot grow from 0 individuals.The fixed points, namely the values of R for which $\frac{dR}{dt} = f(R) = 0$, are classified as either asymptotically stable or unstable based on the sign of the second derivative or given by:

$$\frac{d^2R}{dt^2} = f'(R^*). \tag{4.4}$$

An equilibrium point R is asymptotically stable [26] if all perturbations result in very small deviations which after some time t return to equilibrium. Considering R^* as a fixed-point of continuously differentiable function $f(R)$, where $f'(R^*) \neq 0$. From here $f'(R^*) < 0$ represents an asymptotically stable fixed point R^* and $f'(R^*) > 0$ represents an asymptotically unstable fixed point. Stable fixed points are either sinks or attractors and unstable fixed points are either sources or repellers. In this thesis stable fixed points are classified as attractors and unstable fixed points are classified as repellers.

A complimentary form of steady state or equilibrium analysis is performed by linearising the locally about the equilibrium points of the equation [26]. The procedure is followed as presented in [27].

Considering equation (4.1) with fixed points obtained by solving $f(R) = 0$ for $R = R^*$. Taylor series expansion is performed on $f(R)$ for every fixed point R^* in its local region and this is shown below:

$$f(R) = f(R^*) + \left.\frac{df}{dR}\right|_{R=R^*}(R - R^*) + \frac{1}{2}\left.\frac{d^2f}{dR^2}\right|_{R=R^*}(R - R^*)^2 + ... \tag{4.5}$$

In the local region of R^* all terms of a higher order such as $(R-R^*)$ are insignificant when compared to $(R - R^*)$ [26], hence these terms are excluded from $f(R^*) = 0$ which gives an approximation of $f(R)$ by the function:

$$f(R) = \left.\frac{df}{dR}\right|_{R=R^*}(R - R^*), \tag{4.6}$$

which is simplified by defining parameters $m = \frac{df}{dR}\Big|_{R=R^*}$ as well as $x = R - R^*$ turning equation (4.6) into:

$$\frac{dx}{dt} = \frac{d}{dt}(R - R^*) = \frac{dR}{dt}. \tag{4.7}$$

Therefore equation (4.5) is written as:

$$f(R) = \frac{dx}{dt} = mx, \tag{4.8}$$

which is exponential growth/decay expressed in a linear equation. Hence if $m > 0$ there exists some exponential deviation from the fixed point, which dictates an unstable fixed point. Conversely when $m < 0$ there exists exponential convergence towards the fixed point, dictating a stable fixed point.

Next a two-dimensional system is considered.

4.2.2 Two-Dimensional System

This thesis considers a two-dimensional system, where the following are analysed:

$$\frac{dR_1}{dt} = f(R_1, R_2), \tag{4.9}$$

$$\frac{dR_2}{dt} = g(R_1, R_2). \tag{4.10}$$

In a similar process to one-dimensional systems, linearisation is performed around the fixed points in order to verify the stability of a fixed point in the local region using Taylor's theorem [27].

The phase trajectories are found as solutions of:

$$\frac{dR_1}{dR_2} = \frac{f(R_1, R_2)}{g(R_1, R_2)}, \tag{4.11}$$

meaning that a unique curve is passing through any point (R_1, R_2) excluding the fixed points (R_1^*, R_2^*) for which the following holds true:

$$f(R_1^*, R_2^*) = g(R_1^*, R_2^*) = 0.$$

Performing Taylor series expansion of $f(R_1, R_2)$ and $g(R_1, R_2)$, both of which are nonlinear function, in the local region of the fixed point (R_1^*, R_2^*) results in the following:

$$f(R_1, R_2) = f(R_1^*, R_2^*) + \frac{\partial f}{\partial R_1}\Big|_{R_1^*, R_2^*}(R_1 - R_1^*) + \frac{\partial f}{\partial R_2}\Big|_{R_1^*, R_2^*}(R_2 - R_2^*) + ... \tag{4.12}$$

$$g(R_1, R_2) = g(R_1^*, R_2^*) + \frac{\partial g}{\partial R_1}\Big|_{R_1^*, R_2^*}(R_1 - R_1^*) + \frac{\partial g}{\partial R_2}\Big|_{R_1^*, R_2^*}(R_2 - R_2^*) + ..., \tag{4.13}$$

in which the ellipses represent terms of higher order. These are defined for f as:

$$\frac{1}{2}\frac{\partial^2 R_1}{\partial R_1^2}(R_1 - R_1^*)^2, \tag{4.14}$$

and for g as the following:

$$\frac{1}{2}\frac{\partial^2 R_2}{\partial R_2^2}(R_2 - R_2^*)^2. \tag{4.15}$$

Setting the constants:

$$X = R_1 - R_1^*, \tag{4.16}$$
$$Y = R_2 - R_2^* \tag{4.17}$$

allows for the following simplification:

$$\frac{dX}{dt} = \frac{dR_1}{dt}, \tag{4.18}$$
$$\frac{dY}{dt} = \frac{dR_2}{dt} \tag{4.19}$$

where $f(R_1^*, R_2^*) = 0 = g(R_1^*, R_2^*)$. Therefore the ODEs presented in equations (4.9) and (4.10) turns into the following system:

$$\frac{dX}{dt} = aX + bY + ..., \tag{4.20}$$
$$\frac{dY}{dt} = cX + dY + ..., \tag{4.21}$$

with the constants a, b, c, and d determined by the matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial R_1} & \frac{\partial f}{\partial R_2} \\ \frac{\partial g}{\partial R_1} & \frac{\partial g}{\partial R_2} \end{bmatrix}_{R_1^*, R_2^*}.$$

The matrix $\mathbf{A}$ is known as the Jacobian matrix. The Jacobian uses partial derivatives to show how every component R_i within the system varies relative to all the components including itself. The higher order terms are ignored around the local region of the fixed point, therefore the nonlinear system is approximated by:

$$\frac{dX}{dt} = aX + bY, \tag{4.22}$$
$$\frac{dY}{dt} = cX + dY, \tag{4.23}$$

which is linear. From here the eigenvalues are obtained which offer insight into the vector field geometry around the local region of the fixed points. Eigenvalues are represented by λ_i, and in this case for the two-dimensional system with Jacobian $\mathbf{A}$ the eigenvalues λ_1 and λ_2 are given by computing the determinant of the difference between the Jacobian and the identity matrix multiplied by λ.

$$\mathbf{det}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \lambda^2 - (a+d)\lambda + (ad - bc) = \mathbf{det}\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right). \tag{4.24}$$

From here the eigenvalues $\lambda_{1,2}$ are computed using the quadratic formula:

$$\lambda_{1,2} = \frac{a+d}{2} \pm \frac{\sqrt{(a-d)^2 + 4bc}}{2}. \tag{4.25}$$

Eigenvalues of a matrix show relation to the determinant and trace:

$$\mathbf{det}(\mathbf{A}) = \lambda_1 \lambda_2, \tag{4.26}$$
$$\mathbf{tr}(\mathbf{A}) = \lambda_1 + \lambda_2. \tag{4.27}$$

Therefore the eigenvalues of matrix A are obtained from the determinant and trace using the following form of the quadratic equation:

$$\lambda_{1,2} = \frac{\mathbf{tr(A)} \pm \sqrt{\mathbf{tr(A)} - 4\mathbf{det(A)}}}{2}. \tag{4.28}$$

The eigenvectors associated to these eigenvalues are obtained by solving the following system:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \lambda \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}. \tag{4.29}$$

This concludes the concepts implemented in this thesis. A two-dimensional system is considered and used, which has eigenvalues either both real or complex conjugates, and its simplicity allows for application in population modeling.

The modeling process is explored below.

4.2.3 Modeling Process

Following from the concepts of two-dimensional systems, the factors affecting the rhino-poacher model are discussed below. These include the intrinsic growth rate, carrying capacity and Allee effect as well as exploration of the process of mathematically modeling population growth.

Intrinsic Growth Rate

This thesis considers the growth rate of populations as the intrinsic growth rate, namely the difference between total births and total deaths of the population. Most explanatory works derive the exponential law for the case in which all individuals identical. This mean there is no discrepancy between age, gender, population size or the genetic structure, and reproduction occurs continuously [28]. Below the law of conservation is shown [28], where the number of individuals in a population begins to vary only as a result of birth or deal, or an exit from the system in terms of either immigration or emigration [28]. These rates are then changed to "per capita" [28]:

$$\frac{dR}{dt} = B - C = bR - cR = (b - c)R = aR, \tag{4.30}$$

where the rate of change $\frac{dR}{dt}$ of total number of individuals R with respect to time t. The total birth rate and death rate are given by B and C respectively, while b and c are the per capita rates [28], and a is the per capita rate of population change known as the intrinsic growth rate [28]. The population is assumed closed, therefore there is no representation of immigration or emigration. This is the nature in which a, the intrinsic growth rate, is considered for the remainder of this thesis.

Modeling Growth, Carrying Capacity and Allee Effect

Modeling population dynamics using ODEs shows the rate of change of the considered population with respect to time proportional to the current population. This is known as the growth model, and follows from equation (4.30):

$$\frac{dR}{dt} = aR, \tag{4.31}$$

where $\frac{dR}{dt}$ is the rate of change of population R with respect to time t proportional to population R at intrinsic growth rate a. The population R is a function of time, and is considered at time t where $R(t)$ is non-negative as there cannot exist a negative population. In the case of a positive a the solution shows exponential behaviour. The solution of equation (4.31) is given by:

$$R(t) = Ce^{at},$$

where C is a constant. This shows that the solution to the model presented in equation (4.31) is exponential in nature and continues to grow forever. Some constraints are placed on this model, such as an initial condition.

Initial conditions are seen as the differential equation origin, in this case for the point (R, t), which satisfies the general solution of the ODE. This allows solving for C. For equation (4.31) the initial population R at time $t = 0$ is written as $R(0) = R_0$. Now the solution $R(t)$ is rewritten as:

$$R(t) = R_0 e^{at},$$

where R_0 is the point from which the exponential graph originates. The graph of $R(t)$ is plotted with initial population of rhino $R_0 = 1000$, a growth rate of 50% given by $a = 0.5$ and time $t \in [0, 3]$ in years:

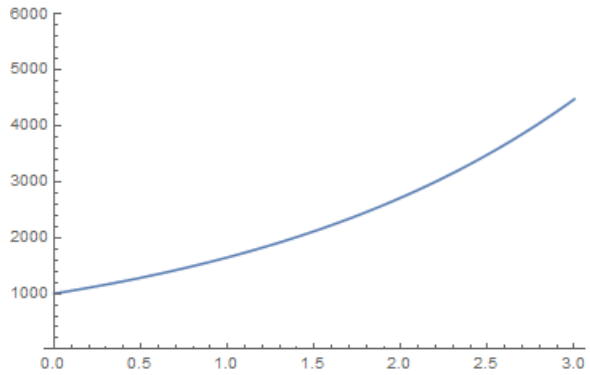


Figure 1: Solution $R(t)$ (blue line) in number of rhino plotted against time t in years.

It is clear from Figure 1 that the population R grows indefinitely, and it is here where the concept of carrying capacity is introduced. In order to prevent the infeasible event of a continuously increasing population this growth is capped by some parameter K which represents the environment's ability to sustain the population under consideration and is a population size beyond which the population cannot go. In order for K to have the desired effect on the model it is formed as follows:

$$\frac{dR}{dt} = aR \times (\alpha),$$

where the term $\alpha \in [0, 1]$ is intended to constrain the exponential increase for large populations as $R \to K$. However for smaller population size $R << K$ the term α places fewer constraints on the system in order for the population to grow freely and tend toward the system in equation (4.31). This is summarised by the logistic growth model:

$$\frac{dR}{dt} = aR\left(1 - \frac{R}{K}\right). \tag{4.32}$$

From equation (4.32) above it is clear that for $R \to K$ the term $\left(1 - \frac{R}{K}\right) \to 0$, and for $R << K$ the term $\left(1 - \frac{R}{K}\right) \to aR$ or the growth model in equation (4.31). The solution to the logistic growth model in equation (4.31) is plotted below. The initial population $R(0) = 10$, the carrying capacity $K = 1000$ and the intrinsic growth rate $a = 0.1$. These parameters are all selected at random to display the general behaviour of the solution, as shown below:

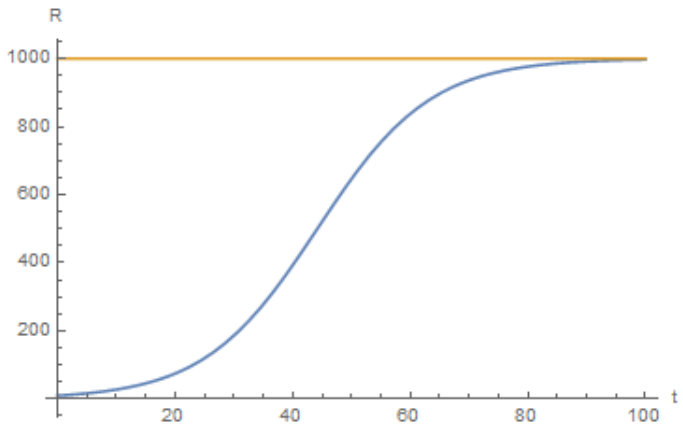


Figure 2: Solution $R(t)$ (blue line) of equation (4.32) in number of rhino plotted against time t in years with carrying capacity (orange line).

The carrying capacity is the value beyond which a population R cannot go, namely K. This limits the population maximum by creating a ceiling. In modeling population dynamics a minimum threshold below which the population R decays is considered. In [29] a clear definition of the Allee effect is not given, but the concept of survival values and the aspects relating to those are explored. The Allee effect describes a scenario in which population density is low and thus causes limitations in the natural population increase factors such as availability of mates as well as limited cooperative behavior [26, 30]. In order to implement the Allee effect into the logistic growth model there needs to exist a minimum boundary for the population above which logistic growth continues, below which the population under consideration tends to extinction [26, 30]. The lower bound set by the Allee effect is also known as the critical point, and is considered in this thesis as the parameter M, below which population tends towards extinction. Considering the logistic growth model in equation (4.32), the Allee effect is represented as follows:

$$\frac{dR}{dt} = aR\left(1 - \frac{R}{K}\right) \times (\beta),$$

where the term β constrains the decaying decrease of small populations as $R \to M$. For larger population size $R >> M$ the term β is positive and imposes fewer constraints on the system displaying dynamics similar in nature to that shown in equation (4.32) as decay due to the Allee effect does not occur under these conditions for population R. However when $R \to M$ the system begins to experience decay and the term $\beta \to 0$, which indicates $\frac{dR}{dt} \to 0$. The common differential equation form of the Allee effect as mentioned in [31] is given below:

$$\frac{dR}{dt} = aR\left(1 - \frac{R}{K}\right)\left(\frac{R}{M} - 1\right), \tag{4.33}$$

where $\frac{dR}{dt}$ is the rate of change of population size R with respect to a, the intrinsic growth rate. The carrying capacity is represented by K and the parameter M is known as the Allee threshold

or the population size below which the system tends to extinction. Considering a population with a large carrying capacity $K = 10000$ with initial population $R(0) = 1500$, critical point $M = 1000$ and intrinsic growth rate $a = 0.03$ the solution to the system in equation (4.33) is shown below:

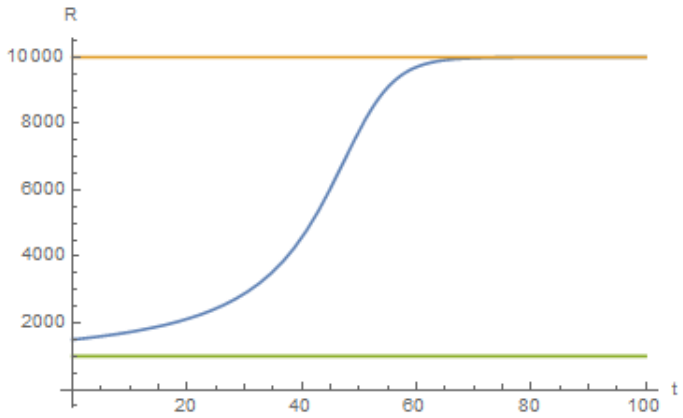


Figure 3: Solution $R(t)$ (blue line) of equation (4.33) in number of rhino plotted against time t in years with carrying capacity K (orange line) and critical point M (green line).

These results are obtained when initial population $R(0)$ is above the Allee effect threshold M. Comparing cases where $R(0) < M$ and $R(0) > M$, with varying intrinsic growth rates a, allows for observation of the effect M has on the solution. Here carrying capacity is set to $K = 1000$, the critical point $M = 250$ and growth rates varied with $a \in [0, 0.9]$. Two values are chosen for $R(0)$, namely $R(0) = 200$ and $R(0) = 280$ and the solutions $R(t)$ are plotted below:

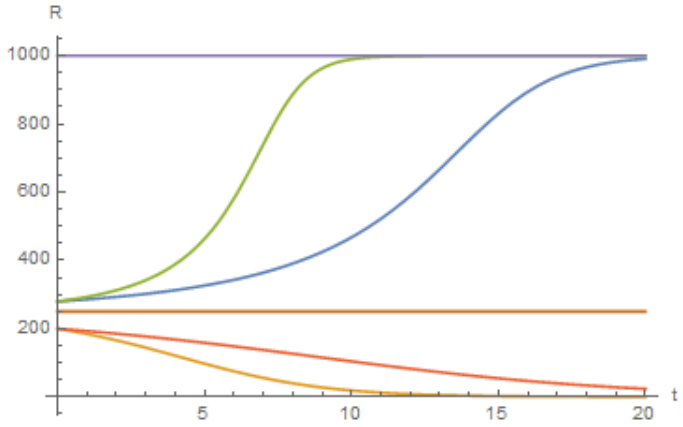


Figure 4: Solutions $R(t)$ of equation (4.33) with population $R(t)$ plotted against time t in years with carrying capacity K (purple line) and critical point M (orange line). Solutions $R(t)$ where $R(0) > M$ (green line, blue line) are above M (orange line) and $R(0) < M$ (yellow line, red line) are below. Steeper increase/decrease in solutions $R(t)$ are governed by intrinsic growth rate a, varied for pairs of $R(t)$. These are defined as $a_{\text{green}} > a_{\text{blue}}$ and $a_{\text{red}} > a_{\text{yellow}}$.

The graph in Figure 4 displays the solution to equation (4.33) with carrying capacity $K = 1000$, critical point $M = 250$ and growth rates varied with $a \in [0, 0.9]$. Each of the four solution curves are based on initial population $R(0)$, and for those curves above M, $R(0) = 280$ and for those below M $R(0) = 200$ in order to illustrate the behaviour of a strong Allee effect [26] where the population R shows negative growth rate for those cases where $0 < R < M$ and tends to extinction whereas a positive growth rate is observed for those population values which satisfy $M < R < K$. The assumption is made that the critical point $M < K$ in all cases, and that M is non-zero. In the presence of a weak Allee effect those populations below M are only affected in their rates of growth [26]. Parameter values chosen for the output in Figure 4.33 are decreased from those in Figure 3. This is because the Allee effect is best observed in small population sizes, as those above M display logistic growth [26].

The modeling assumptions are discussed below.

4.3 Modeling Assumptions

Broadening the scope of a general model is very important, as the goal of this thesis is to present initially an explanatory model which offers insight into, and a fair representation of, the rhino population behaviour while considering the active poacher population estimates and their effect on rhino.

Real-world data and information about certain rates as well as behaviours are combined in order to create a model which displays well all of these factors with the following assumptions:

- A natural growth rate for rhino is assumed as discussed in Section 4.2, and given as $4\% - 6\%$ across multiple sources [2, 32, 33, 34].
- An additional factor of rhino death is assumed in the event of rhino-rhino combat. Black rhino are known to suffer both male and female death due to mortal combat [35] but the figures are not high. The omission of this assumption is, on the large population scale of this thesis, not too noticeable. However it is included for completeness and representation of the real-world situation.
- An Allee effect is assumed as discussed in Section 4.2, making the origin of the system an attracting fixed point. That is, a minimum population is required for rhino persistence.
- The carrying capacity is greater than the minimum rhino population required to sustain the population. This assumption holds true throughout the thesis as the maximum possible number of rhino is always greater than the minimum required to sustain the population.
- Since there is no available data relating to poachers other than arrests as shown in Table 5, P in this thesis represents the active poacher population. It is unknown how many real people are equivalent to one unit of P, however it is known that P represents a number of real people. The unknown in this case is the conversion factor from units of P into number of real people.
- For the active poachers P a rate is chosen to include both the unknown growth rate and the rate at which members of this population exit the system due to external reasons. These include death from either natural causes or rhino-poacher interaction, arrest, illness or disability. This rate is a_2 and considered a catch-all parameter. Population dynamics of P are unknown and it is assumed that the factors leading to P decreasing are greater than the growth rate

of P, from the observed general increase in poacher arrests since 2010 and general decrease in rhino poached since 2014 as shown in Table 5. Therefore a_2 is negative for this stage of the research. This rate is subject to change in the event that both the growth rate and death rate are considered independently, as in Section 4.7.

- Active poachers are assumed to operate in the KNP, public parks and the wild. This research considers the KNP and SA, in accordance with the available poaching data discussed throughout this thesis, as all instances of poaching take place in these areas. No areas outside of those listed here are taken into consideration.
- A natural death rate of active poachers is assumed and made up of either prosecution, unwillingness to continue poaching or death resulting from failed poaching attempts. Those who are no longer poaching rhino are seen as exiting the model. This is part of the rate referred to in this part of the research as a_2.

The nomenclature is given below, listing the definitions of each of the generalised parameters. Parameters are amended for each particular set of differential equations, used to represent the rates for either the rhino or the active poacher population.

	Description
R	The number of rhino still alive and in the system.
P	Units of the active poaching population.
a_1	The rhino population growth rate.
a_2	The active poacher population fluctuation rate.
b	The poaching rate coefficient.
c	The rate at which a population decreases due to reasons other than b.
K	The rhino population's carrying capacity.
M	The Allee effect threshold.
hR	Carrying capacity of active poachers P.

Table 9: Variables and parameters used in this thesis

The rhino population represented by R shows the number of rhino that are alive in the system, in other words it displays the rhino population at time t.

The unit of active poachers in the system at time t is given by P, which is calibrated using the poacher arrest statistics of the poacher arrests from 2010 as shown in Table 5. Comparing the arrest figures and the rhino population figures from Table 6 the ratio of poachers arrested to the total rhino population is roughly 1.5%.

The poaching rate b is an estimate as there are no available data relating to it. The rate c at which members of a population exit the model due to different reasons can represent many unique circumstances. In the case of the rhino, this is attributed to disease and rhino-rhino interaction resulting in death [35]. In the active poacher population this can represent retirement, going into hiding, dying from failed poaching attempts or stopping poaching altogether.

Finally, h is the ratio that dictates the carrying capacity for active poachers as a fraction of rhino numbers given by hR. This rate is unrelated to the h parameter in the Fowler translocation model as presented in equation (3.5). The h used in this thesis is also unrelated to $h(x)$, or harvesting, as presented in [36]. The carrying capacity hR is determined by the remaining rhino population at time t, which indicates a relation between the active poacher and rhino populations. This dictates that the remaining number of rhino at time t affects the active poacher population.

The models are explored below. This thesis begins with the introduction of a simplistic set of Lotka-Volterra differential equations and, incorporating everything covered in this section, iterates towards the models used to obtain results.

4.4 Lotka Volterra Model

The variables and parameters used in this section, for the Lotka Volterra model and its variations, are given:

	Description
R	Rhino population.
P	Active poacher population.
a_1	Natural growth rate of the rhino.
a_2	Active poacher population fluctuation rate.
b_1	Rate at which rhino-poacher interactions result in rhino being poached and exiting the system.
b_2	Rate at which the active poachers population increases due to successful poaching.
K	Carrying capacity, implemented to keep in check the rhino population which does not increase exponentially if the active poacher population becomes either extinct or far less dominant in the model.
M	Minimum number of rhino required to keep the population from extinction.

Table 10: Variables and parameters used in Section 4.4

The following initial system is analysed:

$$\frac{dR}{dt} = a_1R - b_1RP,$$
$$\frac{dP}{dt} = -a_2P + b_2RP.$$

This system is expected to exhibit a correlation between rhino death from poaching and the increase in active poachers. More active poachers are tempted join the population P to kill rhino if this count is high, as it signals a higher success rate. Taking this into consideration, the first variation to the basic Lotka Volterra is given as follows:

$$\frac{dR}{dt} = a_1R - b_1RP, \tag{4.34}$$
$$\frac{dP}{dt} = -a_2P + b_2(b_1RP), \tag{4.35}$$

Here, the rhino-poacher interaction represented by RP decreases the rhino population at rate b_1, and increases the active poacher population at a rate $b_2 \cdot b_1$. This signifies that for every rhino an active poacher successfully poaches, the active poacher population increases due to the increasing success rate of poaching rhino.

If the active poacher population $P \to 0$ the rhino population R is affected only by parameter a_1, and $\frac{dR}{dt} = a_1 R$ is increasing. Inversely, if $R \to 0$, the active poacher population shows a decrease affected by a_2, $\frac{dP}{dt} = -a_2 P$, and become extinct. This follows from the assumption that if there are no rhino, there are no rhino poachers.

The steady state solutions are found by setting both $\frac{dR}{dt}$ and $\frac{dP}{dt}$ to 0. This gives insight into the equilibrium of the model. The above system is simplified for this step as follows:

$$\frac{dR}{dt} = R(a_1 - b_1 P), \tag{4.36}$$

$$\frac{dP}{dt} = P(-a_2 + b_1 b_2 R) \tag{4.37}$$

From here the steady state solutions are found by setting $\frac{dR}{dt}$ and $\frac{dP}{dt}$ to 0.

1. There exists a steady state at $(R^*, P^*) = (0, 0)$; a case in which both populations die out and become extinct.
2. Consider the case $R \neq 0$. This implies that $a_1 - b_1 P^* = 0 \implies P^* = \frac{a_1}{b_1} \neq 0$.
   Similarly for $P \neq 0$. This implies that $-a_2 - b_1 b_2 R^* = 0 \implies R^* = \frac{a_2}{b_1 b_2} \neq 0$.
   Hence the steady state is $(R^*, P^*) = (\frac{a_2}{b_1 b_2}, \frac{a_1}{b_1})$.

Next, the Jacobian matrix of the system is found. This matrix is made up of the system's partial derivatives of the first order [37]. In the case where the Jacobian is square the determinant is called a Jacobian determinant. Here the system of ODEs consists of two variables and two equations which results in a square Jacobian. The Jacobian $J_{i,j}$ consists of partial derivatives with respect to the variables R and P. Rewriting the system from equations (4.36) and (4.37)in matrix form:

$$\begin{bmatrix} R'(t) \\ P'(t) \end{bmatrix} = \begin{bmatrix} R(a_1 - b_1 P) \\ P(-a_2 + b_1 b_2 R). \end{bmatrix}$$

The general Jacobian for this system taking partial derivatives for both variables and equations becomes:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial R}{\partial t} & \frac{\partial P}{\partial t} \\ \frac{\partial R}{\partial t} & \frac{\partial P}{\partial t} \end{bmatrix}$$

The Jacobian for the system under consideration therefore looks like this:

$$\mathbf{J} = \begin{bmatrix} a_1 - b_1 P & -b_1 R \\ b_1 b_2 P & -a_2 + b_1 b_2 R \end{bmatrix}$$

The eigenvalues are calculated after obtaining the Jacobian. Here the goal is to arrive at the characteristic polynomial, which is achieved by setting the determinant $\mathbf{det}(\lambda I_n - J)$ to 0, where I_n is the identity matrix and λ are the eigenvalues relating to Jacobian J in the following way:

$$J\vec{v} = \lambda\vec{v},$$

where the vector $\vec{v}$ is any non-zero vector. This means that λ are the eigenvalues of J if and only if $\mathbf{det}(\lambda I_n - J) = 0$. In the special case that the Jacobian determinant has zeros in all off-diagonals, the values in the diagonals $J_{i,i}$ are the eigenvalues. The steady state solutions are used along with the determinant requirements in the Jacobian to find their corresponding eigenvalues.

1. Considering the equilibrium point $(R, P) = (0, 0)$, the Jacobian becomes:

$$\mathbf{J} = \begin{bmatrix} a_1 & 0 \\ 0 & -a_2 \end{bmatrix}.$$

   Here the matrix only has diagonal values, therefore these are the real eigenvalues of $\lambda_1 = a_1$ and $\lambda_2 = -a_2$.

2. For the equilibrium point $(R, P) = (\frac{a_2}{b_1 b_2}, \frac{a_1}{b_1})$ the Jacobian is given by:

$$\mathbf{J} = \begin{bmatrix} 0 & -\frac{a_2}{b_2} \\ a_1 b_2 & 0 \end{bmatrix},$$

   which does not reduce to simple eigenvalues. In order to find the eigenvalues for this Jacobian, the following determinant is obtained and set to zero:

$$\mathbf{det}(\lambda I_2 - J) = \begin{bmatrix} \lambda & \frac{a_2}{b_2} \\ -a_1 b_2 & \lambda \end{bmatrix} = 0,$$

   which reduces to the following characteristic polynomial:

$$\lambda^2 + a_1 a_2 = 0,$$

   where the eigenvalues are obtained by solving for λ:

$$\begin{aligned} \lambda^2 &= -a_1 a_2 \\ \lambda &= \pm\sqrt{a_1 a_2} \cdot i, \end{aligned}$$

   both of which are imaginary in this case.

The initial model is improved on below.

4.4.1 Introducing Carrying Capacity

Expanding on equations (4.34) and (4.35) the following predator-prey system includes a carrying capacity, K, for the rhino population. This is introduced to remove the assumption that in the absence of active poachers P, the rhino R population grows exponentially:

$$\frac{dR}{dt} = a_1 R\left(1 - \frac{R}{K}\right) - b_1 RP, \tag{4.38}$$

$$\frac{dP}{dt} = -a_2 P + b_1(b_2 RP). \tag{4.39}$$

For an extremely large K, this system reverts to the one in equations (4.34) and (4.35). Therefore it is assumed that K cannot be so large as to cause the system to revert to its earlier state and is defined in Section 1.2.1 as $K = 26000$.

4.4.2 Introducing Minimum Population Parameter

Building onto equations (4.38) and (4.39) this iteration introduces a minimum sustainable population parameter, M, for the rhino population. This is included because of the assumption that the rhino population may reach a population size which implies imminent extinction:

$$\frac{dR}{dt} = a_1 R\left(1 - \frac{R}{K}\right)\left(\frac{R}{M} - 1\right) - b_1 RP, \tag{4.40}$$

$$\frac{dP}{dt} = -a_2 P + b_1(b_2 RP). \tag{4.41}$$

If $\left(1 - \frac{R}{K}\right)\left(\frac{R}{M} - 1\right) \to 1$ then the system presented above reverts to the system in equations (4.34) and (4.35). Since $R < M \implies \frac{dR}{dt} < 0$ means rhino go extinct it is assumed that this is not the case.

The steady state solutions, which offer insight into the equilibrium of the model, are obtained by setting $\frac{dR}{dt}$ and $\frac{dP}{dt}$ to 0. There exist steady states at $(R, P) = (0, 0),\ (K, 0),\ (M, 0)$ and in the case where $p \neq 0$.

1. Consider first the non-trivial case $P \neq 0$. This implies that $-a_2 + b_1 b_2 R = 0 \implies R = \frac{a_2}{b_1 b_2} \neq 0$.
   For $P \neq 0$ the solution becomes $a_1 R\left(1 - \frac{R}{K}\right) \cdot \left(\frac{R}{M} - 1\right) - b_1 RP = 0$. Substituting $R = \frac{a_2}{b_1 b_2}$ this becomes:

$$a_1 \frac{a_2}{b_1 b_2}\left(1 - \frac{a_2}{b_1 b_2 K}\right)\left(\frac{a_2}{b_1 b_2 M} - 1\right) - b_1 \frac{a_2}{b_1 b_2} P = 0,$$

$$a_1\left(1 - \frac{a_2}{b_1 b_2 K}\right)\left(\frac{a_2}{b_1 b_2 M} - 1\right) - b_1 P = 0,$$

$$\text{Therefore; } P = \frac{a_1}{b_1}\left(1 - \frac{a_2}{b_1 b_2 K}\right)\left(\frac{a_2}{b_1 b_2 M} - 1\right).$$

   The assumption is made that $P > 0$, as $R \neq 0$. This entails that either:

   - $1 - \frac{a_2}{b_1 b_2 K} > 0 \implies b_1 b_2 K > a_2$, and
   - $\frac{a_2}{b_1 b_2 M} - 1 > 0 \implies a_2 > b_1 b_2 M$.

   or:

   - $1 - \frac{a_2}{b_1 b_2 K} < 0 \implies a_2 > b_1 b_2 K$, and
   - $\frac{a_2}{b_1 b_2 M} - 1 < 0 \implies b_1 b_2 M > a_2$.

   Hence the steady state is $(R, P) = \left(\frac{a_2}{b_1 b_2}, \frac{a_1}{b_1}\left(1 - \frac{a_2}{b_1 b_2 K}\right) \cdot \left(\frac{a_2}{b_2 b_2 M} - 1\right)\right)$. The Jacobian for the coexistence equilibrium is:

$$\mathbf{J} = \begin{bmatrix} -\frac{b_1 a_2}{b_2} & -\frac{b_1 a_2}{b_2} \\ b_2\left(\frac{a_1}{b_1}\right)\left(1 - \frac{a_2}{b_2 K}\right) & 0 \end{bmatrix}.$$

   The trace of J is $-\frac{a_1 a_2}{b_2 K} < 0$. The determinant of J is:

$$\begin{aligned} \mathbf{detJ} &= \frac{b_1 a_2}{b_2} \cdot \frac{b_2 a_1}{b_1}\left(1 - \frac{a_2}{b_2 K}\right) \\ &= a_2 a_1\left(1 - \frac{a_2}{b_2 K}\right) > 0. \end{aligned}$$

   The eigenvalues of the matrix have negative real parts, indicating a stable fixed point.

2. Considering the equilibrium point $(0, 0)$, the Jacobian is:

$$\mathbf{J} = \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix}.$$

The trace of J is $-a_1 - a_2 < 0$. The determinant of J is:

$$\begin{aligned} \mathbf{detJ} &= (-a_1)(-a_2), \\ &= a_1 a_2 > 0. \end{aligned}$$

The eigenvalues of this matrix also display negative real parts, indicating a stable fixed point.

3. For $(M, 0)$ the following Jacobian is obtained:

$$\mathbf{J} = \begin{bmatrix} a_1\left(1 - \frac{M}{K}\right) & -b_1 M \\ 0 & -a_2 + b_1 b_2 M \end{bmatrix}.$$

The eigenvalues have positive real parts, making this an unstable fixed point.

4. The equilibrium point $(K, 0)$ has Jacobian:

$$\mathbf{J} = \begin{bmatrix} a_1\left(1 - \frac{K}{M}\right) & -b_1 K \\ 0 & -a_2 + b_1 b_2 K \end{bmatrix}.$$

Again there is a positive real part in the eigenvalues of this Jacobian, meaning $(K, 0)$ is an unstable fixed point.

The data-related limitations of the models are discussed below.

4.5 Data Limitations

The lack of data pertaining to rhino poaching in South Africa coupled with the possibility of a large number of local minima for systems in higher dimensions makes poaching difficult to estimate. For this reason some of the parameters in this thesis are fixed, or set. These models are seen as preliminary, and the limited data is explored using ODEs.

These data are used as reference in deciding upon initial parameter ranges wherever applicable, and because of the scarcity of data these ranges themselves are estimates. A portion of parameters are set intuitively within these ranges by manipulation based on how good the fit is between the model solution curve and the available data plot across time t, offering no statistical basis for justifying these values. The remaining parameters are selected using RMS. By setting or fixing a number of parameters, this research becomes limited to the region defined by those set parameters, and explores the remaining parameters within that region. Although a local minimum is obtained, the possibility of other local minima or perhaps even better fits is not explored in this research. There are not enough data to exclude the possibility of other good fits existing in the Euclidean space. This thesis has only explored some of the possible stabilities exhibited by the considered system and acts as a preliminary and an exploratory study into using ODEs to represent dynamic systems and interaction for which there are very limited data. If there were more available and collated data, a better fit could potentially be achieved.

4.6 Rhino-Poacher Model

In order to represent the dynamics of the rhino-poacher interaction this research models both the rhino and active poacher population dynamics.

The active poacher, P, ODE is extended to include a carrying capacity, an Allee Effect minimum population parameter and a rate at which active poachers exit the system. The rate a_2 as presented in Section 4.4 is redefined here. After the active poacher ODE is discussed, the parameter ranges and their justifications are explored before both the SA and KNP rhino-poacher models are proposed.

4.7 Active Poacher Equation

This research considers the unknown general behaviour of P. As there are very limited data relating to poachers outside of poacher arrests as well as number of rhino poached, the factors describing the dynamics of the novel active poacher population P are posited.

The active poachers, P, increase at an intrinsic growth rate a_2 and decrease at an exit rate of c_2 due to external reasons.

An increase rate b_2 based on successful poaching b_1RP dictates the increase of P based on successful poaching.

The active poachers, P are present in the system to poach rhino, hence P is regulated at some population value beyond which active poachers would outnumber rhino, namely a carrying capacity. Since P does not represent exactly the number of individual rhino poachers, the carrying capacity was chosen as a rate of poacher arrests to the rhino population R at time t where the data are collated.

Along with a carrying capacity, the active poacher population has a value below which $P \to 0$. This shows the possibility of active poachers going extinct. It is accounted for with a minimum population parameter, M_2.

In summary, for both the SA and KNP models the active poacher P ODE is given by:

$$\frac{dP}{dt} = a_2 P \left(1 - \frac{P}{hR}\right)\left(\frac{P}{M_2} - 1\right) + b_1 b_2 RP - c_2 P. \tag{4.42}$$

The rhino-poacher system has initial conditions for R relating to the population numbers at time $t = 0$, or the earliest recorded rhino population data used in this thesis. In the SA model $R(0) = 14539$ and in the KNP $R(0) = 10953$. The initial condition for active poacher population $P(0)$ are both chosen as $P(0) = 100$. The parameters used in these rhino-poacher models, excluding those already listed in Table 10, are given below:

	Description
a_2	Active poacher growth rate (redefined from Table 10).
c_1	Rate at which rhino-rhino interactions result in rhino dying and exiting the system [35].
c_2	Rate at which active poachers leave the model due to external reasons.
h	Ratio of R which determines the active poacher carrying capacity.
hR	Active poacher P carrying capacity.
M_1	Allee effect threshold for the rhino population.
M_2	Allee effect threshold for the active poacher population.

Table 11: Parameters used in the SA and KNP rhino-poacher models

The parameter estimation process is explained below.

4.7.1 Parameter Estimation

For this rhino-poacher system there are no known dynamics of the active poacher population P. In order to model the active poachers in this research a novel population P is presented. The active poacher population P interacts with the rhino population R as some form of a predator-prey system, meeting with random chance in a diffusion-like fashion.

Can a meaningful solution be obtained from a system facing all of these limitations? The model solutions produced in this thesis are viewed as constrained projections of the behaviour of rhino population R for time t based on all of the aforementioned parameters and limited data relating to them. When modeling population dynamics with a scarce amount of data as is the case in this research, predictions and stability discussions of the system require certain values of parameters to be set. This is arranged into a two-stage process where parameter ranges and their respective values are justified, estimated, manipulated and optimised and the algorithms used are listed.

The parameter K, carrying capacity, is set at $K = 26000$ and $K = 13800$ for the SA and KNP models respectively in Section 1.2.1 and is the only parameter not open to manipulation or numerical optimisation.

Stage 1: All parameters are open to adjustment (excluding K) within their respective ranges. These ranges are assigned from a combination of the considered literature and educated guesses or estimated values obtained using trial and error. Parameter values are then adjusted by hand across these ranges until an apparently good fit, as judged by eye, is achieved.

Stage 2: A subset of parameters are chosen for optimisation across their respective ranges using RMS, with the remaining parameters set. Parameters are set in order to decrease the dimensions of unknowns in the system and make optimisation of the remaining parameters using RMS possible.

4.7.2 Parameter Ranges

All parameters and their respective ranges are discussed below for both the SA and KNP rhino-poacher models.

SA Model

Here the parameter ranges used in **Stage 1** for the SA model are presented and discussed.

- The general growth rate of rhino was given between $4\% - 6\%$ [2, 32, 33, 34, 38]. This originally places the range at $a_1 \in [0.04, 0.06]$. These are estimated rates, and this research uses floor value of 0.1% to include possible lower growth rates for consideration. The initial range for a_1 is $a1 \in [0.001, 0.06]$.
- The poaching rate b_1 is estimated at 2% for white rhino and 1% for black rhino according to [38]. This research considers initially a higher possible poaching rate, and for this reason the initial range for the poaching rate b_1 is $b_1 \in [0, 0.05]$
- The Allee Effect threshold parameter M_1 for the rhino population R has range $M_1 \in [10, 1000]$ as an educated guess. This range implies that the minimum number of rhino, below which population $R(t)$ tends to extinction, exists between $10 - 1000$ rhino.
- The active poacher growth rate a_2 cannot be quantified and is an estimate. Based on the poaching rate between $1\% - 2\%$ [38] and the positive rhino growth rate [2, 38], this research assumes active poacher growth rate a_2 cannot exceed 3% as an educated guess based on the available data. The same floor value of 0.1% is used and the initial range for a_2 is given by $a_2 \in [0.001, 0.03]$.
- The rate of active poacher increase based on successful poaching, b_1, is unknown. This research considers a low influx or growth of P based on b_1, therefore an educated guess is made to cap this rate at 1%, with a near-zero floor value. The initial range for b_1 is $b_1 \in [0.0001, 0.01]$.
- The rate at which active poachers leave the system due to external reasons, c_2, is also unknown. The only data relating to c_2 is poacher arrests as shown in Table 5. Since very little is deduced from this single data set relating to a unique way in which active poachers are removed from the model, an estimated range is set. As an educated guess, a large rate between $1\% - 30\%$ is chosen, and the initial range for c_2 is $c_2 \in [0.01, 0.3]$.
- The rate of h is some part of the rhino population R which determines the carrying capacity of P. This rate is chosen by comparing umber of poacher arrests as shown in Table 5 to the total rhino population of that year. If the arrests are given by α, then the rate rate $h = \frac{\alpha}{R}$. This value is roughly 1% on average, therefore the initial range for h is $h \in [0.01, 0.02]$
- The Allee Effect threshold parameter M_2 for the active poacher population P is an educated guess and chosen between $1 - 50$ active poachers. The initial range for M_2 is $M_2 \in [1, 50]$.

KNP Model

Here the parameter ranges used in **Stage 1** for the KNP model are presented and discussed.

- The same general growth rate a_1 is considered for the KNP model as for the SA model. However this research considers the possibility of the KNP having no growth rate, $a_1 = 0$. Therefore the initial range for a_1 is $a1 \in [0, 0.06]$.

- The initial range for the poaching rate b_1 is the same as the range considered for the SA model, $b_1 \in [0, 0.05]$
- The rate c_1 at which rhino-rhino interactions, $R \cdot R$ or R^2, result in rhino dying and exiting the system is unknown but prevalent in rhino population [35]. Based on the effect this parameter has on the solution it is chosen to remain very low and take place only in the KNP for the consideration of this research. The initial range for c_2 is $c_2 \in [0.00001, 0.00005]$.
- The Allee Effect threshold parameter M_1 has a lower range for the KNP as an educated guess assuming that less rhino are required to sustain a viable population in a more densely populated area of smaller size. This research considers the minimum number of rhino, below which population $R(t)$ tends to extinction, as between $10 - 300$ rhino. The initial range for M_1 is $M_1 \in [10, 300]$.
- The active poacher growth rate a_2 cannot be quantified for the KNP either and is chosen to remain the same as for the SA model. The initial range for a_2 is given by $a_2 \in [0.001, 0.03]$.
- The rate b_1 is unknown for the KNP as well. The same initial range for b_1 is $b_1 \in [0.0001, 0.01]$.
- The range of c_2 is also unchanged from the SA model, therefore the initial range for c_2 is $c_2 \in [0.01, 0.3]$.
- The range of h remains the same as in the SA model. The initial range for h is $h \in [0.01, 0.02]$
- The parameter M_2 remains between $1 - 50$ active poachers. The initial range for M_2 is hence also $M_2 \in [1, 50]$.

4.8 SA Rhino-Poacher Model

From equations (4.40) and (4.41) the SA model includes a minimum sustainable rhino population parameter, and this is changed from M to M_1 in order to accommodate the variation in the active poacher equation. A carrying capacity hR and a minimum population M_2 for the active poachers P. The active poacher carrying capacity is given by a ratio, h, of the rhino population, R, at time t. It is also assumed that there is some minimum population of active poachers, below which they face imminent extinction. This model considers rhino poaching across South Africa:

$$\frac{dR}{dt} = a_1 R \left(1 - \frac{R}{K}\right)\left(\frac{R}{M_1} - 1\right) - b_1 RP, \tag{4.43}$$

$$\frac{dP}{dt} = a_2 P \left(1 - \frac{P}{hR}\right)\left(\frac{P}{M_2} - 1\right) + b_1 b_2 RP - c_2 P. \tag{4.44}$$

4.9 KNP Rhino-Poacher Model

From the SA model given in equations (4.43) and (4.44) the rate of mortal combat between rhino is represented by c_1 [35]. The KNP model focuses exclusively on rhino poaching within the Kruger National Park and is given by the following set of differential equations:

$$\frac{dR}{dt} = a_1 R \left(1 - \frac{R}{K}\right)\left(\frac{R}{M_1} - 1\right) - b_1 RP - c_1 R^2, \tag{4.45}$$

$$\frac{dP}{dt} = a_2 P \left(1 - \frac{P}{hR}\right)\left(\frac{P}{M_2} - 1\right) + b_1 b_2 RP - c_2 P. \tag{4.46}$$

4.10 Parameter Optimisation

Here the two-stage process as described in Section 4.7.1 is performed.

4.10.1 Stage 1:

The available rhino population data are plotted first for later comparison to the model solution curves. Since the population data are not annualised there are some gaps in times t which are accounted for by connecting the known and available data points. The population data also start at different years, or different times t, for both the SA and KNP models. The rhino population data for the SA model are plotted below:

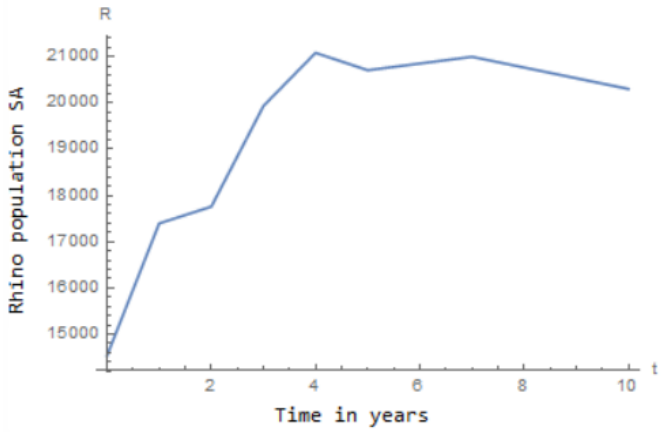


Figure 5: Available rhino population data (blue line) relating to the SA model for time t.

The SA model uses eight data points relating to the eight years for which rhino population numbers are available. These data spanned an 11 year period from $2005 - 2015$ as shown in Table 6. Thus the range considered here for t is $t \in [0, 10]$. The rhino population data for the KNP model are plotted below:

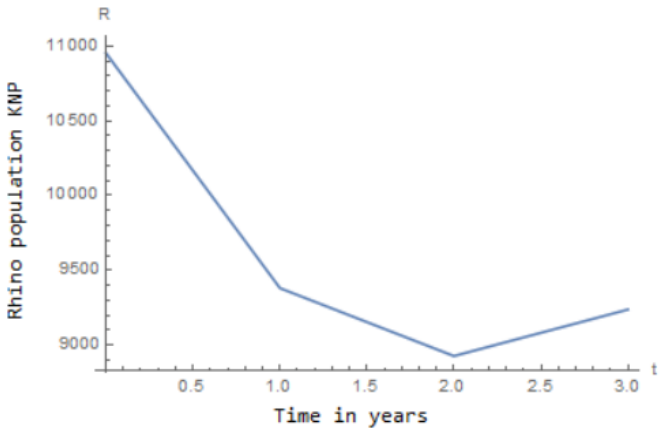


Figure 6: Available rhino population data (blue line) relating to the KNP model for time t.

The KNP model uses four data points relating to the four years for which rhino population figures are available. In this case the data are annualised from $2012-2015$ as shown in Table 7. The range for t in Figure 6 is $t \in [0,3]$ years. The data do not begin at the same t for both models. With the data plots available for both models, their respective systems of ODEs as presented in equations (4.43), (4.44), (4.45) and (4.46) are solved. The function NDSolve is used, which solves differential equations, and this procedure is summarised in the algorithm below:

Algorithm 1 NDSolve

take in $R'(t)$ and $P'(t)$ ODEs as input

set initial conditions $R(0)$ and $P(0)$

declare variables to be solved for, namely R and P

set range for t

solve

Here each parameter is manipulated or iterated toward a set a value within the ranges as set in Section 4.7.2 by hand. The solution curves for the SA and KNP models obtained using NDSolve are compared to their respective raw data curves in Figures 5 and 6. The variations in solutions and the comparisons to the raw data that follow are made by varying parameter values across their initial ranges using the Manipulate function, as shown below:

Algorithm 2 Manipulate

take in **Algorithm 1** NDSolve as input

plot graph of $R(t)$ and $P(t)$ evaluated at their respective solutions

set range for t

declare range of values to be plotted on $y-$axis

include plot of available rhino population data

declare and list unknown parameters

set lower boundary of the parameter range

set upper boundary of the parameter range

set parameter range step size

output solution plots based on parameter values chosen from range

Manipulation is performed by hand, moving the parameter values across their initial ranges using sliders until the resulting solution curve resembled closely the general shape of the data curve. Other than K, all parameters are open to adjustment in this stage.

This research uses the concept of "good fits" and "poor fits". A good fit is equivalent to a solution curve obtained from parameter manipulation across initial ranges which resembles in shape and general trajectory the raw data curve, for times t where this comparison is possible. A poor fit is considered the opposite. Both good and poor fits are attainable by parameter variation, and

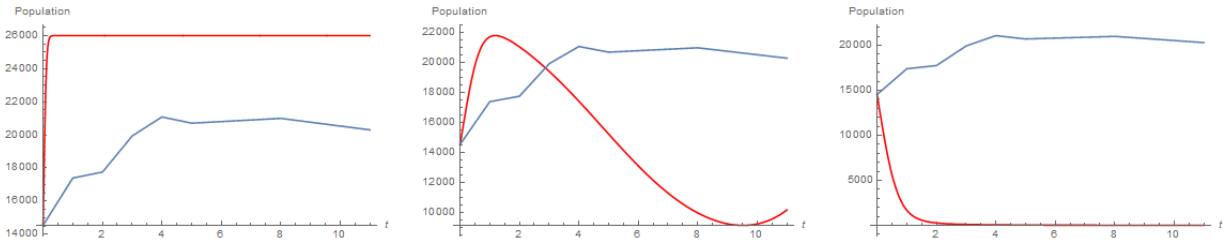
these are explored below for both the SA and KNP models. The initial parameter ranges as presented in Section 4.7.2 are repeated below:

	SA Model	**KNP Model**
Rhino Equation Parameters		
a_1	$[0.001; 0.06]$	$[0; 0.06]$
b_1	$[0; 0.05]$	$[0; 0.05]$
c_1	–	$[0.00001; 0.0005]$
M_1	$[10; 1000]$	$[10; 300]$
Active Poacher Equation Parameters		
a_2	$[0.001; 0.03]$	$[0.001; 0.03]$
b_2	$[0.0001; 0.01]$	$[0.0001; 0.01]$
c_2	$[0.01; 0.3]$	$[0.01; 0.3]$
h	$[0.001; 0.03]$	$[0.001; 0.03]$
M_2	$[1; 50]$	$[1; 50]$

Table 12: Initial parameter ranges chosen for manipulation as discussed in Section 4.7.2

It is in these initial ranges that the manipulation of parameter values is performed. The following interpolations occur between the first rhino population data point at $t = 0$, and the final time t for which this data are released, namely $t = 11$ and $t = 3$ for the SA and KNP models respectively. Three cases of poor fits are shown first to highlight the possibility of these fits existing within the initial ranges. Following the poor fits, a good fit is shown to confirm one is possible to achieve within the initial parameter ranges.

For the SA model, both the good and poor fits are shown below:



(a) All parameters at lowest range value

(b) All parameters at randomised values

(c) All parameters in the middle of their respective ranges

Figure 7: SA model solution $R(t)$ (red line) with population R plotted against time t as poor fits in comparison to data (blue line).

From these poor fits it is clear further manipulation across the parameter ranges is required in order to achieve a good fit, as shown below:

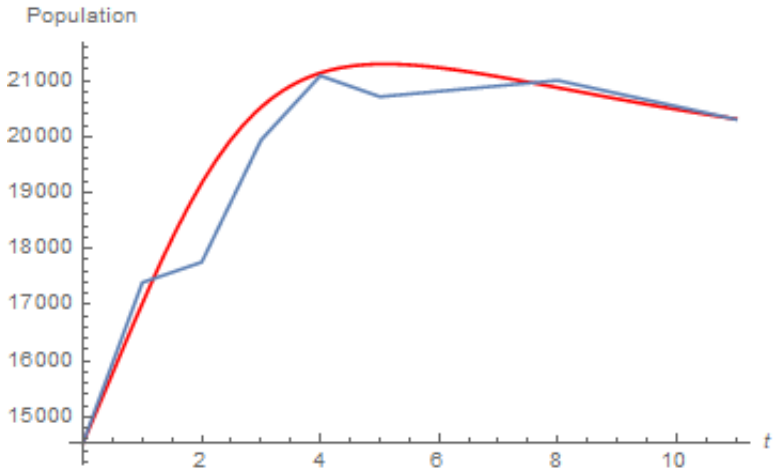
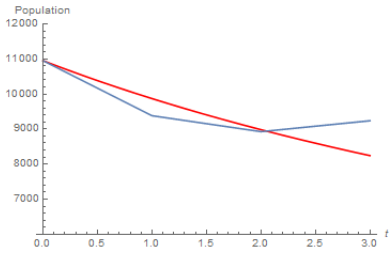


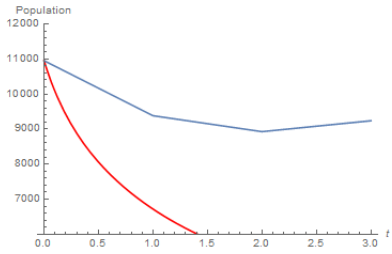
Figure 8: SA model solution $R(t)$ (red line) with population R plotted against time t as good fit in comparison to data (blue line).

The solution curve in Figure 8 above confirms that within the SA model initial parameter ranges introduced in Section 4.7.2 and collated in Table 12 there exists a good fit solution. This is considered a local minimum.

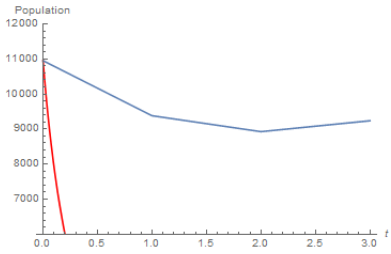
For the KNP model, the solutions which showed good and poor fits to the available data are presented below:



(a) All parameters at lowest range value



(b) All parameters at randomised values



(c) All parameters about the middle of their respective ranges

Figure 9: KNP model solution $R(t)$ (red line) with population R plotted against time t as poor fits in comparison to data (blue line).

Considering these poor fits for the KNP model solution curves it is clear that further parameter value manipulation across the initial ranges is necessary to achieve a good fit, as shown below:

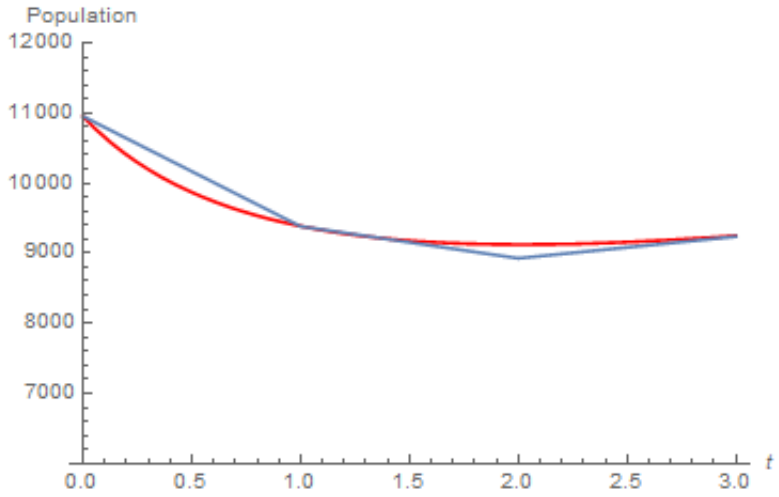


Figure 10: KNP model solution $R(t)$ (red line) with population R plotted against time t as good fit in comparison to data (blue line).

The solution curve in Figure 10 above confirms that within the KNP model initial parameter ranges there exists a good fit solution. Again, this is considered a local minimum.

4.10.2 Stage 2

A subset of parameters are set to constant values along their respective initial ranges. The remaining parameters are numerically optimised using RMS along their redefined ranges. These ranges are chosen based on the effects their respective parameters have on the solution curve; the ranges within which the solution curve did not experience fluctuations so great that they affected the goodness of the fit.

Set Parameters

The parameters a_1, b_1, c_1, M_1 and M_2 have been set based on the solutions produced from manipulation across their initial ranges by hand as well as educated guesses. Each of the set parameter values produced good fit solutions.

The Allee Effect threshold parameters M_1 and M_2 are assigned values first as the minimum population is a prerequisite to producing solutions for the rhino-poacher system. For simplicity the active poacher population threshold is set at $M_2 = 10$ for both the SA and KNP models. In the case of the threshold M_1 for the rhino population it is set at $M_1 = 900$ for the SA model and $M_1 = 100$ for the KNP model. The SA model has a larger M_1 as the entire South African rhino population is considered, including that of the KNP.

The intrinsic growth rate a_1 for the SA model is considered across the range $a_1 \in [0.04, 0.06]$ as confirmed in several sources [2, 32, 33, 34, 38]. The solution curve with the best fit is obtained at $a_1 = 0.055$, which is within the dictated range. For the KNP model manipulation is performed across the initial range $a_1 \in [0, 0.06]$ as there are no data relating to KNP growth rates. The best fit solution is achieved at $a_1 = 0.015$.

Manipulation of the rate c_1 across the initial range $c_1 \in [0.00001, 0.00005]$ fluctuates considerably the solution curve. The value $c_1 = 2 \times 10^5$ produces a good fit solution for the KNP model. This parameter does not appear in the SA model.

The poaching rate b_1 produces a good fit solution curve at $b_1 = 0.002$ for the SA model, and this parameter is kept the same for the KNP model.

These set parameters and their respective values are summarised below:

	SA Model	**KNP Model**
K	26000	13800
M_1	900	100
M_2	10	10
a_1	0.055	0.015
b_1	0.002	0.0045
c_1	-	2×10^{-5}

Table 13: Parameters manipulated to set values

Both SA and KNP models are now 4-dimensional systems, with 4 parameters remaining for numerical optimisation within their ranges.

Remaining Parameters

The remaining parameters are a_2, b_2, c_2 and h. Each of these relates to the active poacher population P and is assigned a redefined range, justified below, within which RMS is performed.

The rate h of R dictates the active poacher P carrying capacity. The number of poachers arrested, as listed in Table 5, and the rhino population numbers in Table 6 are compared at equal time t. The ratio of rhino to arrested poachers is $\in [0.008, 0.015]$. The range of h is therefore between $1\% - 2\%$ and set as $h \in [0.01; 0.02]$ for both the SA and KNP model.

The only data relating to the rate at which active poachers leave the system c_2 are the poacher arrests as listed in Table 5. This research considers this rate between $6\% - 15\%$. Within the range $c_2 \in [0.06, 0.15]$ solution curves show a good fit to the raw data curve for the SA model. The KNP is considered more secure than South Africa in terms of protecting rhino, therefore the rate c_2 for the KNP model is larger than for the SA model. The range $c_2 \in [0.1, 0.25]$ provides good fit solution curves in comparison with the raw data curves.

There are no available data relating to the intrinsic growth rate a_2 of active poachers. The ranges of a_2 for the SA and KNP models are therefore educated guesses. For the SA model the range is $a_2 \in [0.008, 0.02]$, and $a_2 \in [0.008, 0.015]$ for the KNP model. These are educated guesses represent a slow growing active poacher population, which grows slower in the KNP.

The rate b_2 is considered last. No educated guess is made, the range is chosen based purely on the resulting solution curves and their respective fits to the raw data curves. Good fits are achieved for both the SA and KNP models where $b_2 \in [0.001, 0.002]$.

In summary the ranges considered for the remaining parameters are:

	SA Model	**KNP Model**
a_2	$[0.008; 0.02]$	$[0.008; 0.015]$
b_2	$[0.001; 0.002]$	$[0.001; 0.002]$
c_2	$[0.06; 0.15]$	$[0.1; 0.25]$
h	$[0.01; 0.02]$	$[0.01; 0.02]$

Table 14: Remaining parameter ranges

These parameter ranges are seen as the constraints placed on the system; the 4 unknown dimensions. With the constraints and their ranges set, the SA and KNP models are rewritten to include the set parameter values. The SA model is presented below:

$$\frac{dR}{dt} = 0.055R\left(1 - \frac{R}{26000}\right)\left(\frac{R}{900} - 1\right) - 0.002RP, \tag{4.47}$$

$$\frac{dP}{dt} = a_2P\left(1 - \frac{P}{hR}\right)\left(\frac{P}{10} - 1\right) + 0.002b_2RP - c_2P. \tag{4.48}$$

The KNP model is shown below:

$$\frac{dR}{dt} = 0.015R\left(1 - \frac{R}{13800}\right)\left(\frac{R}{100} - 1\right) - 0.0045RP - 0.00002R^2, \tag{4.49}$$

$$\frac{dP}{dt} = a_2P\left(1 - \frac{P}{hR}\right)\left(\frac{P}{10} - 1\right) + 0.045b_2RP - c_2P. \tag{4.50}$$

For both models the parameters which are not set are left as constants, and these parameters are optimised numerically through error calculation using RMS.

Error Calculation

Root Mean Square (RMS) errors are calculated in terms of the solution points at time t compared to the raw data points at equal times t. The solutions for the SA and KNP models are obtained by using the error-minimising parameter values from the ranges presented in **Stage 2**. In order to locate these error-minimising values within the parameter space the error is calculated with RMS.

The RMS takes the square root of the difference between the raw rhino population data and that given by the solution and squares this difference. For the raw data given by $\bar{R}$ and the model solution data by R, the sum of differences squared is given by $\sum(\bar{R} - R)^2$. Following this the RMS considers the square root of the average of these squared errors divided by the number of data points N, and this is shown below:

$$\sqrt{\frac{\sum(\bar{R} - R)^2}{N}}.$$

The error calculating process used in this thesis is summarised in the pseudocode:

Algorithm 3 RMS

take in unknown parameters as input

perform **Algorithm 1** NDSolve

store solution for R(t)

square root of square of difference between available rhino data and model

solution R(t) at time t

sum the error values for each time t where data available

RETURN total RMS value

Once the RMS is calculated, the model solutions, RMS values and parameter values relating to those are stored in a table. The table is then searched through to find the lowest resulting RMS value along with the associated parameters. It is these lowest RMS-value parameters which are referred to in this research as "error-minimising" parameters. These error-minimising parameters are obtained by iterating through their ranges with set discretisation or step size. The pseudocode for this process is given below:

Algorithm 4 Error Minimisation

create table to store parameter and RMS data

take in unknown parameters and **Algorithm 3** RMS as input

declare parameter ranges and discretisation

perform **Algorithm 3** RMS across set ranges with declared step size

store RMS values obtained for each parameter value considered

RETURN lowest RMS value and corresponding parameter values

The chosen ranges and their respective step size are summarised below:

	SA Model		**KNP Model**	
	Range	Step Size	Range	Step Size
a_2	[0.008 : 0.02]	0.001	[0.008 : 0.015]	0.0005
b_2	[0.001 : 0.002]	0.0002	[0.001 : 0.002]	0.00001
c_2	[0.06 : 0.15]	0.001	[0.1 : 0.25]	0.0005
h	[0.01 : 0.02]	0.0002	[0.01 : 0.02]	0.0001

Table 15: Parameter ranges with respective step size

At every step the solution value is calculated and compared to the available and corresponding rhino population figures at time t. For the SA rhino-poacher model this is given by $\sqrt{\frac{\sum(\bar{R}-R)^2}{8}}$ and for the KNP rhino-poacher model this is given by $\sqrt{\frac{\sum(\bar{R}-R)^2}{4}}$. For both models the lowest RMS value is returned, along with the parameter values within the range which minimised the error. These are the chosen error-minimising parameter values.

4.11 Error-Minimised Parameter Values

The error-minimising parameter values obtained from RMS for both the SA and KNP models are:

	SA Model	**KNP Model**
a_2	0.008	0.008
b_2	0.001	0.001
c_2	0.06	0.25
h	0.0154	0.01

Table 16: Error-minimised parameter values for the SA and KNP models

These error-minimised parameter values determine the solutions of both the SA and KNP models, which are presented in the following section.

For comparison, several solutions are plotted below against the available data plot where three of the parameter values from Table 16 are kept constant while the remaining parameter is changed to its boundary value(s).

Initially the rate c_2 at which active poachers leave the system and decrease P is considered. For the SA model the solution for the upper bound value $c_2 = 0.15$ is:

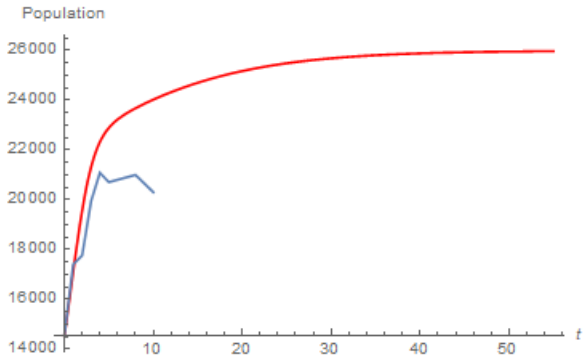


Figure 11: SA model upper bound c_2 solution $R(t)$ (red line) with population R plotted against time t as a poor fit in comparison to data (blue line).

Figure 11 shows a poor fit solution. Increasing the rate at which active poachers P leave the system alters the solution and the prediction of the behaviour of R. In this case, rhino reach carrying capacity $K = 26000$.

For the KNP model the solution obtained for value $c_2 = 0.1$ is:

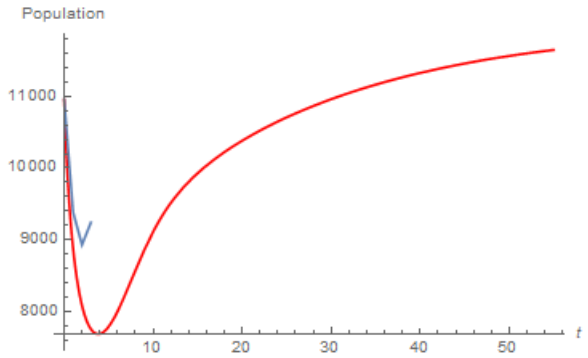
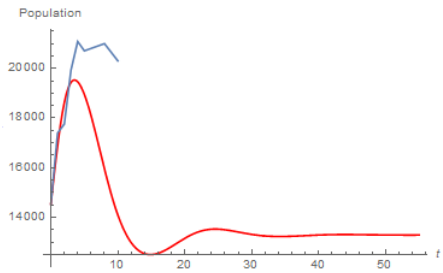


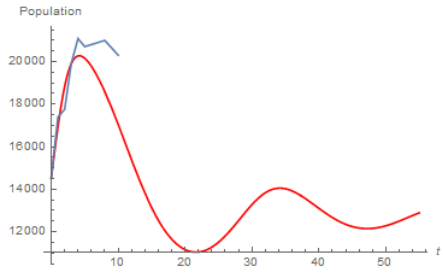
Figure 12: KNP model upper bound c_2 solution $R(t)$ (red line) with population R plotted against time t as a poor fit in comparison to data (blue line).

The solution in Figure 12 is also a poor fit. Decreasing c_2 for the KNP model decreases the population R for initial times t. Population dynamics exhibited by R are no longer in line with the available data.

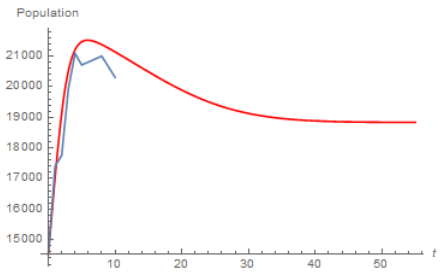
Comparison is performed between available data and solution plots for the remaining rates a_2, b_2 and h at their boundary values. The boundary parameter value solution plots for the SA model are:



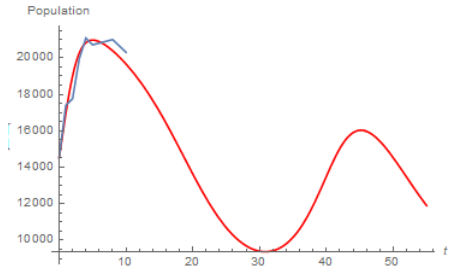
(a) $a_2 = 0.02$ (upper bound)



(b) $b_2 = 0.002$ (upper bound)



(c) $h = 0.01$ (lower bound)

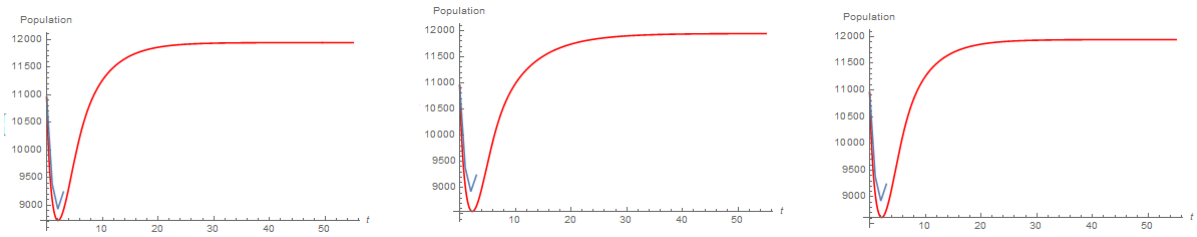


(d) $h = 0.02$ (upper bound)

Figure 13: SA model parameter boundary solutions $R(t)$ (red line) with population R plotted against time t in comparison to data (blue line).

From Figure 13 it is clear that increasing the active poacher P growth rate, a_2, skews the solution away from the available data curve and results in a poor fit. The upper bound of rate of increase in P due to successful poaching, b_2, produces a solution in which rhino population R fluctuates noticeably across time t. However, this solution is a poor fit to the available data. At the lower bound of h, which dictates the active poacher P carrying capacity, the population R tends to stabilisation at roughly $R = 19000$. This solution is also a poor fit. Contrary to this, the upper bound of h produces a seemingly good fitting solution in which fluctuation in R is even more distinct than for the upper bound of b_2. Although this may seem a good fit, a better fit is found using RMS and the error-minimised value of h is found near the middle of its range as shown in Table 16.

The boundary parameter value solution plots for the KNP model are:



(a) $a_2 = 0.015$ (upper bound) (b) $b_2 = 0.002$ (upper bound) (c) $h = 0.01$ (lower bound)

Figure 14: KNP model parameter boundary solutions $R(t)$ (red line) with population R plotted against time t in comparison to data (blue line).

There is little variation in the general shape of the boundary parameter solution curves for the KNP model shown in Figure 14. The active poacher P growth rate, a_2, at its upper bound results in a slightly poorer fit than the solution presented in Section 5, using the error-minimised parameter values. For the successful poaching increase rate b_2 at its upper bound the solution curve shows a poor fit, similarly to the plot of the lower bound of h.

These parameter boundary solutions for both the SA and KNP models show the behaviour of the solution curves as a single error-minimised parameter is varied to one (or both) of its bounds while the others remain fixed. The resulting plots show scenarios which are not considered in this research, as the best fitting solutions are achieved with the error-minimised parameter values. The results in Figures 13 and 14 show the possibility of population dynamics which are different from those considered in this research. The solution space of this research is constrained, several parameters are set prior to numerical optimisation and the final solutions presented in Section 5 are considered local minima.

The model limitations are explored below.

4.12 Model Limitations

The limitations faced in the modeling process of this thesis are discussed below.

Small pool of data

The data regarding rhino poaching are incredibly scarce, and when it comes to data relating to poachers in South Africa the only available figures are those of reported poacher arrests as shown in Table 5. Rhino enclosures and national parks do not release constant updates on rhino population figures, or any data relating to rhino due to concerns of attracting poaching. The values provided for annual rhino populations are gathered and shared by various NGOs, private and public rhino enclosures as well as ecologists and scientific professionals annually as presented in Section 1.2. These population figures are all, of course, estimates as counting methods in several parts of South Africa are very basic and have a large collection of possible mistakes and miscount due to observer error in most cases. There is no way of testing the validity of the data used, adding to the problems created by how few data points are available to begin with. The results in this thesis exist locally in a constrained system, and are constrained predictions.

No structure to active poacher population

There is no available poacher data aside from police arrests made in connection to poaching for the years 2010-2017. The problem with this data set is that it offers very little in the way of broadening the scope and understanding of rhino poaching or the behaviour or makeup of the poacher population, as the poachers are the main predator of rhino.

Using only these figures along with rhino population data from year to year yields an estimated number of active poachers based on the numbers of poachers arrested, number of rhino poached and various estimates of poaching success rates. Due to this lack of data, and a general lack of data relating to most aspects of rhino population, this thesis is forced to create an estimated representation of the active poacher population. This decision is made as the models are designed to analyse and predict rhino population variation, which cannot be performed without one of the main rhino mortality factors: poachers. The active poachers are then represented in the same way as the rhino, showing their population change over time. It is this unobserved active poacher population that provides the difficulties for this research. However, even in this novel population there predictions are made about the projected behaviour of P, which can be tested indirectly.

Therefore, this model provides an estimate of active poachers per year relating to the estimates of rhino population per year. Modelling the poachers as the predators in the predator-prey system is novel, and any methods of tracking this population give rise to several problems. When compared to the rhino data and its observation methods, poachers are incredibly difficult to keep a track of in the same way. This still ignores the fact that poachers themselves are rarely the only ones involved in the process, as the end users of the rhino horn remain far down the supply chain which forms a system structure that is currently impossible to model. Therefore the active poacher population suggested in this thesis is thought of as more of an abstract representation of the potential that exists for poaching, rather than a real, tangible collective of people who prepare, hunt and poach rhino.

Active poacher carrying capacity

Due to the nature of data used in this research, both estimated populations are restricted from exponential growth by a carrying capacity. In this research active poachers are seen as a single entity in a two-entity (predator-prey) closed system. As no external factors are taken into consideration, the estimate chosen as the active poacher, P, carrying capacity is some factor, h, of the current rhino population R. The carrying capacity hR therefore fluctuates based on the rhino population R at time t. In a closed system with limited available data a carrying capacity of one estimated population P is dependent on some ratio h of another estimated population R. However, as the rhino population data are obtained regularly and seen as reliable estimates, and poacher activity remains a challenge to model, this approach is chosen for the carrying capacity.

Ignored mechanisms

A number of important mechanisms relating to the rhino-poacher system which currently exist are not explicitly modeled in this thesis. These mechanisms are excluded due to the added complexity they bring to the model with so many unknown factors already under consideration.

The model predictions can be wrong based on any of the points raised in Section 4.12, including the exclusion of important mechanisms. For example seasonality is a factor which affects all populations in some way and is not accounted for in this thesis. This exclusion is made despite the availability of weather and climate tracking methods because the rhino data available related only to time t in years and no density-related data, nor information on the location of rhino, it is difficult to find a way to relate these factors in this research. Modeling weather- or climate-related population dynamics is therefore not explored in this thesis.

This thesis has assumed both the rhino and active poacher populations as constant in the sense that no distinction is made between age, gender and location of individuals making up either of these populations. With the limited data already a limitation for this thesis, there are currently no ways to track and study the most basic poacher population behaviours, let alone its dynamics, which is why in this thesis the active poacher population P is introduced as a hidden variable. If data regarding the structure of rhino or poacher populations were made available it would be possible to split the populations up based on these data, namely age, sex, gender and location. These data would give insight to population density as well as the probability of a group of rhino R increasing by some number of rhino x based on the available rhino within the considered community, their age and reproductive ability.

The model presented in this thesis is a two-dimensional system of ODEs modeling the poacher rhino interaction dynamics as a predator-prey system. Because of this, only these two populations are considered. In reality, however, there are a large number of other populations present which have an impact on both the rhino and active poacher populations. These include plants or other food sources for the rhino to eat, where available food is a population that has an immediate effect on any population which depends on it for survival. This is not considered because the available data offered no indication of how vegetation and available food resources affect the rhino, as well as where, when and how often these populations interact.

A population which potentially affects both the rhino and the active poachers are anti-poaching units. This includes private security, animals within close proximity to rhino and the police to name a few. This thesis originally considered modeling a third population of security, namely S, but with the number of unknowns only increasing this idea was disregarded. However, that is not to say it is not valid. On the contrary, modeling the dynamics between rhino-poacher-security offers another applicable restriction to the active poacher population and adds more substance to the estimated population P. Modeling anti-poaching dynamics is difficult due to the lack of data, or any consistency in the way that this population interacts with P. Inclusion of an anti-poaching population in this thesis was decided against, as the dimensions of the problem along with unknown parameters are increased. If any data, perhaps in more focused smaller rhino habitats, were made available this population could be modeled well, but that is not the case.

Finally the end-product consumers of rhino poaching are excluded from this thesis, namely the buyers of rhino horn. This population is excluded due to the lack of relating data, since buying rhino horn is performed through several back channels and illegal trades. With no way to model this population, it can be considered rather as a rate. The rate at which rhino poaching increases due to customer demand. This is an unknown quantity and requires estimation, adding to the complexity and dimensions of the system.

Parameter estimates

With some parameters set and others optimised numerically as discussed in Section 4.7.1 the solution space within which the results of this model exist is dictated by the values assigned to these parameters. As parameters are initially set, this creates a constrained system, removing a number of dimensions but also imposing a number of constraints. The ranges chosen for the parameters under numerical optimisation as listed in Table 14 dictate the boundaries of the system. These limit the freedom of the parameters. With the set parameters assigned fixed values, the remaining parameters are optimised based on those values assigned to the set parameters. Because of this even the numerical optimisation takes place in a constrained solution space which is dependent on those values given to the parameters which are fixed. Hence the predictions are seen as constrained, and the minimum obtained is assumed a local minimum. In summary there is no guarantee that the method implemented in this thesis has found the best possible values. These results are only one set of predictions occurring in one instance of the modeled system, an instance decided upon based on the values of the unknown parameters. The results from this thesis can be tested as they are predictions about future rhino population, and these can be compared to the future figures when they are made available.

5 Results

The results from the models described in Sections 4.8 and 4.9 is presented and discussed here. Solutions to both the rhino and active poacher populations, as well as their respective figures, as shown in this Section are all obtained using the NDSolve function, The error-minimising parameter values for the SA model are:

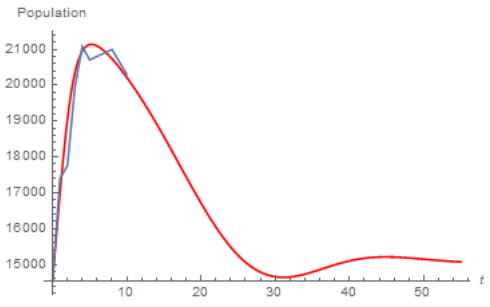
$$a_2 = 0.008,\ b_2 = 0.001,\ c_2 = 0.06,\ h = 0.0154,$$

For the SA model three of the four parameters are on their lower bounds with only h taking a value within its range. In the system considered in this thesis, the parameters with lower bound values are a_2, the active poacher intrinsic growth rate, $b_2 = 0.001$ the rate at which successful poaching increases P and c_2, the rate at which active poachers exit the system. As stated in Section 4.10.1 the ranges assigned to these parameters in Table 14 represent the system under consideration in this thesis. The active poacher intrinsic growth rate $a_2 = 0.008$ indicates that the active poacher population P grows intrinsically at the slowest possible rate within this system. The ratio h is close to the lowest range value at $h = 0.0154$, meaning that the active poacher population carrying capacity is close to its median value in the system. The rate at which successful poaching $0.002RP$ increases the active poacher population is given by $b_2 = 0.001$ and shows that successful poaching attracts the least possible active poachers into the model for this system. Finally, the rate at which active poachers leave the model due to factors other than those already covered in the model is given by $c_2 = 0.06$, meaning the least possible active poachers leave this system due to unaccounted for factors. It is these error-minimised parameter values that lead to the SA model which has been fitted to the data:

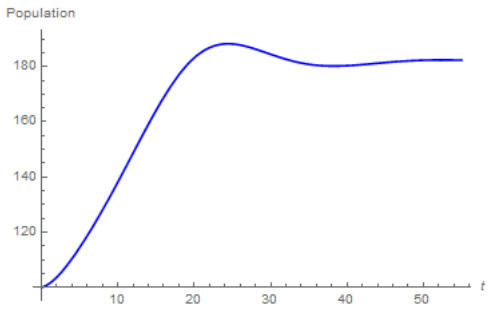
$$\frac{dR}{dt} = 0.055R\left(1 - \frac{R}{26000}\right)\left(\frac{R}{900} - 1\right) - 0.002RP, \tag{5.1}$$

$$\frac{dP}{dt} = 0.008P\left(1 - \frac{P}{0.0154R}\right)\left(\frac{P}{10} - 1\right) + 0.002 * 0.001RP - 0.06P. \tag{5.2}$$

These solutions are considered for time $t \in [0, 55]$ to show predictions made by the model. These predictions are observed along the solution curve outside the initial range for t, in which the available data are plotted to show the goodness of fit. This allows for clear observation of the extrapolation:



(a) Solution $R(t)$ (red line) in number of rhino for the SA model in comparison to available data plotted against time t



(b) Solution $P(t)$ (blue line) in estimated number of active poachers for the SA model plotted against time t

Figure 15: Solutions $R(t)$ (red line) in rhino numbers and $P(t)$ (blue line) in active poacher estimates for the SA model fitted to the data plotted against time t.

For the KNP model, the error-minimising parameter values are:

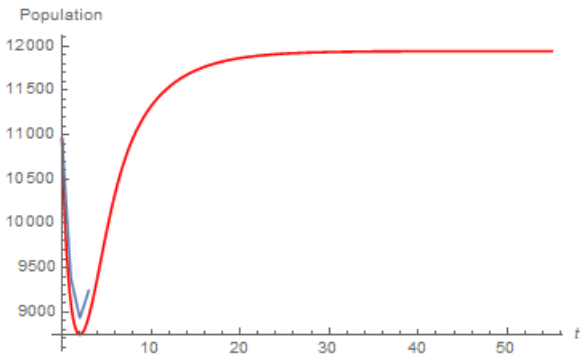
$$a_2 = 0.008,\ b_2 = 0.001,\ c_2 = 0.25,\ h = 0.01,$$

The parameters numerically optimised using RMS lie within ranges presented in Table 14, and as discussed in Section 4.10.1 these ranges are the constraints placed on the system under consideration in this thesis. For the KNP model three of the four parameters are at their lower boundary, while the rate at which poachers leave the system, c_2, is at its upper boundary. The rates a_2, b_2 and h relate to active poacher increase and carrying capacity. The intrinsic growth rate $a_2 = 0.008$ exhibits the slowest possible growth for P within this system, while $b_2 = 0.001$ indicates the lowest possible increase in P as attributed to successful poaching. For $h = 0.1$ this means that active poacher population P has the lowest possible carrying capacity as a rate of rhino population R within the system considered in this thesis. These parameter values complete the KNP model fitted to the data:

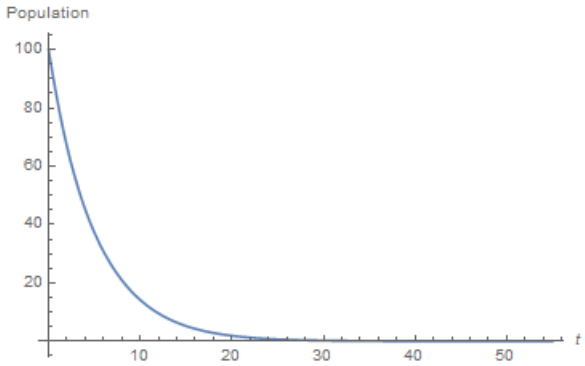
$$\frac{dR}{dt} = 0.015R\left(1 - \frac{R}{13800}\right)\left(\frac{R}{100} - 1\right) - 0.0045RP - 0.00002R^2, \tag{5.3}$$

$$\frac{dP}{dt} = 0.008P\left(1 - \frac{P}{0.01R}\right)\left(\frac{P}{10} - 1\right) + 0.0045 * 0.001RP - 0.25P. \tag{5.4}$$

These solutions are also considered for time $t \in [0, 55]$ to show predictions made by the model. The predictions are again observed along the solution curve outside the initial range for t, where the available data plot shows the goodness of fit. The extrapolation is shown below:



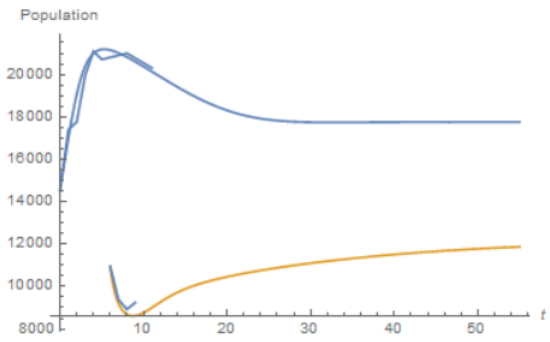
(a) Solution $R(t)$ (red line) in number of rhino for KNP model in comparison to available data plotted against time t



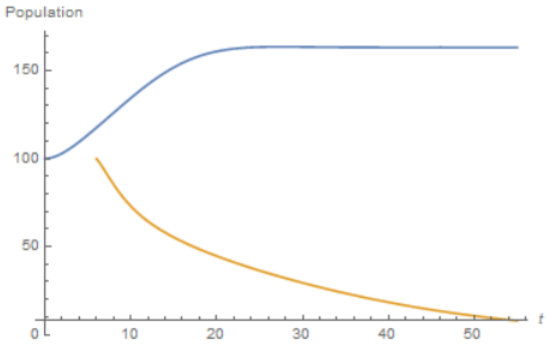
(b) Solution $P(t)$ (blue line) in estimated number of active poachers for the KNP model plotted against time t

Figure 16: Solutions $R(t)$ (red line) in rhino numbers and $P(t)$ (blue line) in active poacher estimates for the KNP model fitted to the data plotted against time t.

The solution curves of both models are plotted on the same axes:



(a) Solutions $R(t)$ in number of rhino for the SA (blue line) and KNP (yellow line) models in comparison to available data plotted against time t



(b) Solutions $P(t)$ in estimated number of active poachers for the SA (blue line) and KNP (yellow line) models plotted against time t

Figure 17: Final solutions $R(t)$ in rhino numbers and $P(t)$ in estimated active poachers for both the SA (blue) and KNP (orange) models fitted to the data plotted against time t.

5.1 Stability Analysis

The models fitted to the data are analysed separately below. Initially the model Jacobian is presented, followed by the fixed points. After this the eigenvalues of each point are shown and thus each fixed point's classification is discussed.

5.1.1 SA Rhino-Poacher Model

Using the numerical values for all parameters in the SA rhino-poacher model the following Jacobian is obtained:

$$\begin{bmatrix} J_{1,1} & -0.002R \\ \frac{0.519487\left(\frac{P}{10}-1\right)P^2}{R^2} + 2\times 10^{-6}P & J_{2,2} \end{bmatrix},$$

where the diagonal values $J_{i,i}$ are presented below:

$$J_{1,1} = 0.55\left(\frac{R}{900}-1\right)\left(1-\frac{R}{26000}\right) + 0.000611111R\left(1-\frac{R}{26000}\right) - 0.002P - 0.0000211538\left(\frac{R}{900}-1\right)R,$$

$$J_{2,2} = 0.008\left(1-\frac{64.9351P}{R}\right)P - \frac{0.519487\left(\frac{P}{10}-1\right)P}{R} + 0.008\left(\frac{P}{10}-1\right)\left(1-\frac{64.9351P}{R}\right) + 2\times 10^{-6}R - 0.06.$$

The real fixed points obtained numerically are shown below:

1. $(R \to 0,\ P \to 0)$
2. $(R \to 900,\ P \to 0)$,
3. $(R \to 14827.5,\ P \to 182.869.)$,
4. $(R \to 25204.8,\ P \to 22.713)$,
5. $(R \to 26000,\ P \to 0)$

Each of these cases are used to find and evaluate the eigenvalues. All values of R and P are rounded to the nearest integer.

Case 1: $(R \to 0,\ P \to 0)$

This fixed point represents extinction for both populations. The eigenvalues are $(-0.08, -0.055)$ are both negative, indicating this is a stable fixed point.

Case 2: $(R \to 900,\ P \to 0)$

This fixed point represents the case in the system where the rhino population tends towards its minimum possible population size and the active poacher population tends to extinction, namely $(R \to M_1, P \to 0)$. This case has eigenvalues $(0.530962, -0.0662)$. Since the two eigenvalues have different signs, this fixed point is classified as a saddle.

Case 3: $(R \to 14828,\ P \to 183)$

This fixed point represents the an equilibrium within the system where the rhino population tends towards 14828 and the active poacher population tends towards 183. In this instance both populations coexist. This case has eigenvalues $(-0.0960142, -0.0844227)$. In this case both eigenvalues are negative hence the fixed point is stable.

Case 4: $(R \to 25205,\ P \to 23)$

This fixed point represents the an equilibrium within the system where the rhino population tends towards general stability close to its carrying capacity and the active poacher population tends towards 23. In this instance both populations coexist. This case has eigenvalues $(-1.39276, 0.0164969)$. Since the two eigenvalues have different signs, this fixed point is classified as a saddle.

Case 5: $(R \to 26000,\ P \to 0)$

This fixed point represents the case in the system where the rhino population tends towards its carrying capacity and the active poacher population tends extinction, namely $(R \to K, P \to 0)$. This case has eigenvalues $(-1.53389, -0.016)$. Since both eigenvalues are negative, this fixed point is classified as stable and shows the properties of an attractor.

The feasible fixed points are classified and tabulated below:

Fixed Point	Classification
$(R \to 0,\ P \to 0)$	Attractor
$(R \to 900,\ P \to 0)$	Saddle
$(R \to 14828,\ P \to 183)$	Attractor
$(R \to 25205,\ P \to 23)$	Saddle
$(R \to 26000,\ P \to 0)$	Attractor

Table 17: SA model feasible fixed points and respective classification

There are three strict attractors within this system, namely the fixed points $(R \to 0,\ P \to 0)$ $(R \to 14828,\ P \to 183)$ and $(R \to 26000,\ P \to 0)$. This indicates three possible instances: both the rhino and active poacher populations achieve extinction, the active poacher population goes extinct while the rhino population reaches carrying capacity, both the rhino and active poacher populations coexist close to their carrying capacities (this is attributed to parameters b_1 and b_2). The system consists of five feasible fixed points, three of which are strict attractors. There are three possible long-term outcomes for this system: extinction of both populations, coexistence of rhino and active poachers, and rhino only. Which of these three states is attained in the system is dictated by the initial conditions. It is for this reason that the system shows tri-stability.

5.1.2 KNP Rhino-Poacher Model

Using the numerical values for all parameters in the KNP rhino-poacher model the following Jacobian is obtained:

$$\begin{bmatrix} J_{1,1} & -0.0045R \\ \frac{1.5\left(\frac{P}{10}-1\right)P^2}{(R+1)^2} + 9\times10^{-6}P & J_{2,2} \end{bmatrix}.$$

The values of the diagonals $J_{i,i}$ are given below:

$$J_{1,1} = 0.015\left(\frac{R}{90}-1\right)\left(1-\frac{R}{13800}\right) + 0.000166667R\left(1-\frac{R}{13800}\right) - 0.0045P - 1.08696\times10^{-6}\left(\frac{R}{90}-1\right)R - 0.00004R,$$

$$J_{2,2} = 0.0015\left(1-\frac{100.P}{R+1}\right)P - \frac{1.5\left(\frac{P}{10}-1\right)P}{R+1} + 9\times10^{-6}R + 0.015\left(\frac{P}{10}-1\right)\left(1-\frac{100.P}{R+1}\right) - 0.165.$$

The real fixed points obtained numerically are shown below:

1. $(R \to 0,\ P \to 0)$,
2. $(R \to 102.377,\ P \to 0)$,
3. $(R \to 10774.7 - 1795.27i,\ ,\ P \to 47.5155\ + 44.8839i)$,
4. $(R \to 10774.7 + 1795.27i,\ ,\ P \to 47.5155\ - 44.8839i)$,
5. $(R \to 12131.6,\ P \to 0)$.

As in the SA model, the eigenvalues are obtained and evaluated from these cases. All values of R and P are rounded to the nearest integer.

Case 1: $(R \to 0,\ P \to 0)$

This fixed point represents the case of rhino and active poacher extinction. Here both of the eigenvalues $(-0.164, -0.015)$ have negative signs, indicating a stable fixed point.

Case 2: $(R \to 103, P \to 0)$

This fixed point represents the case in the system where the rhino population tends towards extinction and the active poacher population tends to its minimum possible population size. The eigenvalues are $(-0.179079, 0.0148734)$, and since the two eigenvalues have different signs this indicates a saddle point.

Case 3: $(R \to 12132, P \to 0)$

This fixed point represents an instance of the system where the rhino population tends towards stable population of 12132 rhino and the active poacher population tends to extinction. This case has eigenvalues $(-1.76248, -0.0708156)$, both of which are negative indicating this fixed point is stable.

The feasible fixed points are classified and tabulated below:

Fixed Point	**Classification**
$(R \to 0,\ P \to 0)$	Attractor
$(R \to 103,\ P \to 0)$	Saddle
$(R \to 12132,\ P \to 0)$	Attractor

Table 18: KNP model feasible fixed points and respective classification

The two strict attractors within this system are the fixed points $(R \to 0,\ P \to 0)$ and $(R \to 12132, P \to 0)$. In both cases the active poacher population goes extinct, whereas the rhino population either goes extinct or tends towards its carrying capacity. The system consists of three feasible fixed points, two of which are strict attractors. There are two possible long-term outcomes for this system: either both populations go extinct or the rhino population stabilises at carrying capacity and the active poachers go extinct. It is the initial conditions that dictate what the outcome, or state, is for this system as is the case for the SA rhino-poacher model in Section 5.1.1. The system therefore exhibits bi-stability.

5.2 Results Discussion

This research models poaching as the main source of rhino population decrease using data collected intermittently over a period of 15 years and shows the predicted behaviour of both rhino R and active poacher P populations for time t. The SA model considers data pertaining to rhino R and active poachers P across South Africa, while the KNP model uses data relating exclusively to the Kruger National Park. Solutions of the SA model fitted to the data show that the rhino and active poacher populations both persist and enter co-existence within the constrained system. The KNP model solutions predict the active poacher population goes extinct while the rhino population stabilises within the constrained system. These results may be subject to overfitting as several unknown parameters are considered and very few data points are available for comparison.

The solutions to the rhino-poacher ODEs presented in this research are dependent on the parameter values obtained in Stages 1 and 2. Initially for the SA model there are 8 unknown parameters in total, and 9 for the KNP model. In **Stage 1**, 4 parameters are set for the SA model and 5 parameters are set for the KNP model. This decreases the dimensions of the problem but restricts the solution space. All solutions attainable outside of the set parameter values are excluded from consideration in this research. In **Stage 2** the remaining 4 unknown parameters for each model are assigned a range and all solutions for parameter values outside of these ranges are excluded from consideration as well. The solution space becomes more constrained with these restrictions. These parameters are numerically optimised using RMS across their respective ranges and the resulting parameter values are considered âerror-minimisedâ. As shown in Section 4.11 there exist solutions within the parameter ranges, for parameter values which are not error-minimised, that exhibit different behaviour to the final results as shown in Figure 17 . Due to several constraints placed upon this system, the results presented in this research are viewed as local minima, rather than global minima. Solutions shown in Figure 13 confirm the possibility of different future scenarios for populations R and P, or other local minima, within the already constrained system considered in this research. Other possible local minima may exhibit vastly different behaviour to the results obtained and presented in Section 5. Although the results obtained in this thesis are predictions, they are constrained. Hence these results are viewed as a local minimum found within a system based on the dynamics of a two-population predator-prey system in which active poachers are the predator and rhino the prey based on the values attributed to unknown parameters. This thesis has shown that using a slightly unorthodox approach the results, or predictions, are obtained as estimates which exhibit overall rhino population stabilisation across South Africa and the extinction of active poachers in the KNP.

The extrapolation of both SA and KNP model solutions in itself is uncertain. The nature of these results, therefore, is also viewed as uncertain. Neither of the predicted states may ever be achieved, and the possibility of rhino poaching leading to rhino extinction is still very real. Despite that fact, however, the obtained predictions of possible stable coexistence are intriguing.

6 Conclusions

The fixed point analysis shows both systems have one strict attractor in which the active poacher population tends to extinction, while the SA rhino-poacher model also has an equilibrium state as a strict attractor. In the case of the active-poacher-free equilibrium state the SA model's solution tends to the fixed point $(R \to 26000, P \to 0)$, and for the the KNP model this fixed point is $(R \to 12132, P \to 0)$. The SA fixed point of equilibrium tends to $(R \to 14828, P \to 183)$ where both populations coexist.

The chosen ranges for the set parameters are based on available data as listed in Section 1.2. Due to the varying figures across sources, these ranges are introduced to consider the space within which a parameter value is found relating to a local minimum. If the figures relating to the parameters considered in this research explicitly known the dimensions of this model would be greatly decreased making not only the analysis simpler, but the results far more reliable due to the number of unknowns decreasing. Besides these sets of data, the poacher population is a complete unknown. In theory, every person can be part of the poacher population, but simply never perform the act of poaching. For this reason the thesis considers active poachers, to visualise the predicted poacher

population involved in poaching rhino. These population figures are chosen based on the small amount of available poacher arrest data relating to the loss of rhino per year. This thesis has attempted to conceptualise a set of active poachers. The behaviour of this suggested population is, therefore, both dependent on and depended on by the rhino population. An increase in active poachers has a decreasing impact on the rhino population. Considering this the rhino population is affected by the active poachers but not the active poachers alone. The parameters used within the thesis help create a wider narrative in terms of what affects the rhino population, and how. Although the results in this thesis show a stabilised population of rhino for both models, real populations are never static.

In this thesis two short but best available time series of rhino population data are obtained, collated and used in an attempt to parameterise a dynamic population predator-prey model on an MSc level. The model is formed to represent the general parameters and rates which affect the rhino population R, and in turn a similar approach is taken to creating a model representing the estimated population of active poachers P. The growth rates of both models are considered as intrinsic, and both a carrying capacity and an Allee effect threshold is implemented for both populations. The parameters or rates for which there are relating data are set within these bounds based on any information relating to the parameter or rate being considered. This approach is taken to account for the number of unknowns and decrease the dimensions and complexity of the system being numerically optimised. The poaching intensity in this thesis is modeled as the result of a population of active poachers and this estimated population's carrying capacity is related to a rate of the rhino population. In this way the small amounts of available data are used to create an intuitive estimate of the potential population of active poachers which exhibits behaviours and trends that make sense when considering how a population of poachers behaves. The stability analysis is performed on fixed points of the model, excluding bifurcation analysis.

The main limitations of this thesis stem from the paucity of data, as highlighted by the data presented in Sections 1.2 and 1.3, and as discussed in Section 4.6. Normally when modeling population dynamics the system under consideration offers sufficient data to reduce the number of unknowns in the chosen model and therefore make for simpler mathematical computation in lower dimension spaces. Because of the limited data available, the results obtained in this thesis are seen as constrained, however scenarios are presented to show the possible use of such models applied to the problem of rhino poaching.

The solution for the SA model fitted to the data in Figure 17 and the stability analysis performed dictate that $R(t)$ and $P(t)$ coexist in equilibrium indefinitely across t. In the case of the KNP model fitted to the data, the solution in Figure 17 and the following stability analysis dictate that $R(t)$ stabilises across t and $P(t) \to 0$. Figure 17 show one excursion for the rhino populations in both the SA and KNP models, after which they stabilise indefinitely.

The parameters and their assigned values and ranges have placed a number of constraints or restrictions on the proposed system. With most of the optimised parameter values found at the boundary of their respective ranges, it is clear that better fits or other local minima exist outside of these bounds. However, these bounds relate to a hidden population of active poachers on which there are no significantly observable or measurable data. It is from these results error-minimising

parameter values that conclusions are draw not only for the rhino population, but also the active poacher population. As both populations are modeled such that they exhibit a clear relationship, the predictions made for the rhino population R as time t increases are based on an assumed population P. This thesis considers active poacher population P as a dynamic system defined and constrained by the ranges presented in Table 14. These parameters describe the assumed behaviour of a population which cannot be observed.

For the SA model the error-minimised parameters relating to active poacher population fluctuation are found at their lower bounds. This indicates a system in which the rate of increase in $P(t)$ is low, but so too is the rate at which individuals exit the system. This thesis has therefore presented in the SA model predictions of the rhino population $R(t)$ in South Africa based on a population $P(t)$ which exhibits low rates of population variation across time t. This population behaviour is shown in Figures 15 and 17. The results in these figures predict stabilisation and coexistence of populations R and P within the SA model as presented in this research. These predicted scenarios are estimated solutions produced within the constrained system.

In conclusion this thesis has proposed a system of differential equations and shown that this approach is applicable to a modeled system representing the interaction dynamics between rhino and poachers with limited or no data pertaining to either of these populations. This thesis is considered an initial and exploratory work, and the solutions obtained predict future population behaviour for both the rhino and active poacher populations within a system based on limited data and the constraints placed on the model.

7 Further Model Improvements

This thesis has initiated a possibility and showcased a minimal demonstration of feasibility within the suggested system. The predicted rhino population and active poacher population curves are only estimates, and are treated as such, however the means in which they are obtained can be used to further the research within this specific mathematical scope relating to modeling population dynamics on systems for which there is very little available data.

The factors excluded in this thesis can be considered, such as rhino gender and age as well as the density of rhino within particular parts of South Africa or the KNP allowing for the research to consider the population as not constant but rather made up of unique groups of rhino. Annual poaching and rhino population survey data would allow for more accurate interpolation as the current sets of available data are inconsistent. The rate at which the climate changes, as well as the effects these changes have on a population, are not considered. These can include the impact floods, droughts, natural disasters and extreme changes in temperature have on the population being considered. Insight into any of these factors allows for further examination of the specific, measurable external forces affecting the system, which allow for more concise conclusions to be drawn from the predictions. These include insight into what the average age and dominant gender of poached rhino is, where these rhino are being poached and where rhino are finding both a safe and providing environment as well as where they are finding it difficult to survive without the intervention of poachers.

The parameter ranges considered in this research can be extended, or different parameter ranges altogether can be explored. These could produce unique results, or other local minima, and allow for comparison of various parameter space solutions. In this expanded solution space unique scenarios could arise if the set parameters were initially assigned different values or ranges, or if numerical optimisation was performed on all the unknown parameters.

More extensive data relating to poachers would be greatly beneficial to this thesis. Other than the arrest figures of poachers in Table 5 there is no quantifiable or measurable figure which describes the behaviour of the poacher population. This is due to many factors, including the fact that there is no way of knowing how large the poacher population is in the real world, therefore there is nothing to base an assumption off of relating to this population. External data on poachers, such as attempted poaching events and their outcome along with the times and areas where rhino-poacher interaction took place, offer valuable insight into the population referred to in this thesis as P. If the poaching data mentioned above are obtained from a smaller rhino environment, such as a national park, the model is able to produce more accurate predictions when decreasing its dimensions. Access to data relating to these factors decreases the dimensions of the problem, making both analysis and optimisation a simpler process.

The formulation process and boundaries of hidden population P can be expanded. This thesis considered P within the assigned boundaries given in Table 4.10.1 in order to obtain error-minimised parameter values relating to this population. These parameters are found mostly on the range boundaries, and expansion of these in future work with justifiable adjustments to the ranges are achievable if more data become available relating to poachers.

Anti-poaching measures can also be explored, including a factor which dictates that at some rate x the chance of a poacher succeeding in poaching a rhino is decreased. This can occur in the event of private security, traps and recording equipment. These exist in some form or another in the real-world system, and inclusion of these factors broadens the scope of consideration of this thesis.

There are other approaches to modeling population dynamics. The methodology used in [17], namely the systems of translocation models, represented the population at time $t + 1$ as the population at time t plus some rate r of the population. This is also bounded above by a carrying capacity as used in this thesis, and then there is some decrease h in the population which models translocation. Here the population is predicted based on the population from the previous year and done so iteratively.

This thesis did not consider stochastic nor spatial approaches but there exist several models and methods which do consider dynamics of a spatial nature, namely agent-based models [39] which model interaction dynamics of unique individuals or groups, known as agents, in order to highlight the effect these agents have on the entire system. Partial differential equations (PDEs) are implemented in ecology studies and research because they explain spatial dynamics of the system and the variation this has on populations [40]. Difference equations are popular in population modeling because they do not assume continuous births as noted in [41]. These are used to model populations with discrete birth pulses, and are also easier to solve and construct [41].

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